

PRÜFER ALGEBRAIC SPACES

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ABSTRACT. This is the first in a series of two papers concerned with relative birational geometry of algebraic spaces. In this paper, we study Prüfer pairs of algebraic spaces that generalize spectra of Prüfer rings, and pushouts of algebraic spaces that generalize composition of valuations and Ferrand's pincements of schemes. As a particular case of Prüfer spaces we introduce valuation algebraic spaces, and use them to establish valuative criteria of properness and separatedness that sharpen the standard criteria. In the sequel paper, we will introduce a version of Riemann-Zariski spaces (*RZ spaces*), and will prove Nagata compactification theorem for algebraic spaces.

1. INTRODUCTION

1.1. Motivation. The main aims of this paper are to study Prüfer algebraic spaces that naturally generalize spectra of Prüfer rings, and pushouts of algebraic spaces that naturally generalize composition of valuations and Ferrand's pincements of schemes. As a particular case of Prüfer spaces we introduce valuation algebraic spaces, and use them to establish valuative criteria of properness and separatedness that sharpen the standard criteria.

This is the first in a series of two papers concerned with generalizing the results and methods of [Tem3] to algebraic spaces. In particular, in the sequel paper [TT] we will introduce a version of Riemann-Zariski spaces (*RZ spaces*), and will prove Nagata compactification theorem for algebraic spaces. We hope that the same method will apply with minor changes to representable morphisms between stacks, and may also be useful in studying non-representable morphisms, but this is planned to be our future joint research with D. Rydh. Our current aim is to sharpen the method of [Tem3] in the relatively simple context of algebraic spaces.

Valuation rings and their spectra, that we call *valuation schemes*, play an important role in the theory of RZ spaces in general, and in [Tem3] in particular. One reason for this is that valuation schemes are atomic objects for the topology of modifications of integral schemes, namely valuation schemes are precisely the integral local schemes that do not possess non-trivial modifications. In order to extend the method of [Tem3], we study the analogous objects in the category of algebraic spaces. In particular, we use *valuation algebraic spaces* to establish valuative criteria of properness and separatedness that refine the usual criteria with valuation schemes, and we establish for them certain pushout constructions that generalize the usual composition of valuations. Both tasks could be done in a rather ad hoc manner, similarly to [Tem3, §3.2] and [Tem3, §2.3], respectively. However, in our research we discovered that most of results concerning valuation algebraic spaces and their pushouts hold more generally for the classes of Prüfer algebraic spaces

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and Ferrand's pushouts, as we explain below. Both objects seem to be interesting on their own, so we devote the entire paper to their study far beyond what will be used in [TT].

1.2. The overview. In §2, we recall basic facts about algebraic spaces, and study pro-open immersions. In the quasi-compact case, pro-open immersions are nothing but flat monomorphisms, which were studied by Raynaud [Ray] in the category of schemes. We extend his results to the category of algebraic spaces. The key result says that pro-open immersions are schematic, hence we do not get anything new when the target is a scheme. Surprisingly, we found no simple proof of this fact, and had to use difficult approximation results from [Ryd1].

In [Fer], Ferrand studied pushouts of schemes, that he called pincements in French. These are pushouts of the form $Y \sqcup_T Z$ where $T \rightarrow Y$ is affine and $T \hookrightarrow Z$ is a closed immersion. In the case when $T \rightarrow Y$ is a flat schematically dominant monomorphism, Rydh generalized Ferrand's results to stacks with quasi-affine diagonal, see [Ryd3, §6]. The goal of §3 is to generalize the results of Ferrand to algebraic spaces without making any assumption on $T \rightarrow Y$ except affineness. In particular, we show that Ferrand's pushouts of schemes are also pushouts in the category of all algebraic spaces. The main results of this section are Theorems 3.3.3 and 3.3.7 showing, in particular, that if Ferrand's pushout datum $(T; Y, Z)$ admits an affine presentation then it possesses a pushout $X = Y \sqcup_T Z$ in the category of all algebraic spaces, and establish many natural properties of these pushouts, including descent of various properties.

In §4 we introduce *Prüfer algebraic spaces* as finite disjoint unions of integral qcqs spaces that do not admit non-trivial modifications. Our main result about the structure of such spaces is Corollary 4.1.6. In particular, it asserts that for an integral qcqs algebraic space X , the following conditions are equivalent: (a) X is Prüfer, (b) X does not admit non-trivial blow ups, (c) some (hence any) affine presentation of X is the spectrum of a Prüfer ring. Our theory of Prüfer spaces may be viewed as the geometric generalization of the classical theory of Prüfer rings. For reasons that at first may look technical, it is desirable to have some control on the modification loci for non-Prüfer spaces. This leads us to the following generalization: Assume that X is a qcqs algebraic space with a subset $U \subset |X|$ closed under generalizations. The pair (X, U) is called Prüfer if X does not admit modifications that are isomorphisms over U . It turns out that almost all results about Prüfer spaces can be generalized to pairs, and the work with pairs even simplifies some proofs. Therefore, almost all results of §4 are proved for pairs. In addition, we prove that U underlies a pro-open subspace with an affine pro-open immersion morphism $U \hookrightarrow X$ and it turns out that if $X = \operatorname{Spec}(A)$ is affine then $U = \operatorname{Spec}(B)$ and (B, A) is a Prüfer pair of rings, a well known notion in abstract commutative algebra. In particular, when X is local this simply means that (B, A) is a semivaluation ring in the sense of [Tem3].

Finally, we study valuation algebraic spaces in §5. These are Prüfer algebraic spaces with a single closed point. We show that if a valuation space X is separated then it is the spectrum of a valuation ring, and give various examples of non-separated (hence non-schematic) valuation spaces. In §5.2, we establish Zariski-local valuative criteria of separatedness, universal closedness, and properness, in which one does not extend the fraction field of the valuation. Note that in order to achieve Zariski-locality it is necessary to deal with all valuation algebraic spaces,

including the non-separated ones. Finally, we prove a stronger criterion of properness in Theorem 5.2.4. This is an analog of [Tem3, Prop. 3.2.3], and this is the criterion we will need in [TT].

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2. PRELIMINARIES

2.1. Algebraic spaces. A basic reference to algebraic spaces is [Knu]. The underlying topological space of an algebraic space X will be often denoted by $|X|$. We say that X is *local* if $|X|$ is quasi-compact and has a unique closed point (without quasi-compactness, the set of closed points can be empty). All interesting results will be proved for *qcqs* (i.e. quasi-compact and quasi-separated) algebraic spaces, so the reader can always assume that all geometric objects are qcqs.

2.1.1. Presentations. If not said to the contrary, (*resp. affine*) *presentation* of a scheme or an algebraic space X means an (*resp. affine*) étale presentation, i.e. a surjective étale morphism $X_0 \rightarrow X$ whose source is a (*resp. affine*) scheme. To give a presentation is equivalent to give the étale equivalence relation $X_1 \rightrightarrows X_0$ with an isomorphism $X_0/X_1 \xrightarrow{\sim} X$, so the latter will often be used to refer to the presentation. If X is qcqs then we automatically consider only its quasi-compact presentations, so the word quasi-compact will usually be omitted.

2.1.2. Zariski points. An algebraic space η is a *point* if any monomorphism to η is an isomorphism. It is well known that any point is the spectrum of a field. A *point* of an algebraic space X is a morphism $f: \eta \rightarrow X$ from a point. If f is a monomorphism then we say that f is a *Zariski point*. There is a natural bijection between points $x \in |X|$ and isomorphism classes of Zariski points $\mathrm{Spec}(K) \rightarrow X$,

and one says that $k(x) := K$ is the residue field of x . We will often use the following two facts, for whose proof we refer to [Knu, Ch 2, Prop. 6.2 and Th. 6.4]: any point $f: \eta \rightarrow X$ uniquely factors through a Zariski point $\eta_0 \rightarrow X$, any point of an algebraic space X factors through some étale presentation $X' \rightarrow X$.

2.1.3. Limits and pushouts. By *limits* (resp. *colimits*) we will mean projective (resp. injective) limits. We will say that a pushout (resp. a limit) *exists* in a category \mathfrak{C} if it is representable by an object X of \mathfrak{C} . In such case, we will freely say that X is the pushout (resp. the limit) although, strictly speaking, it is only defined up to a unique isomorphism.

2.1.4. Approximation. All algebraic spaces in this section are assumed to be *qcqs*. Throughout the paper we will use various results of what we call *approximation theory*, that studies filtered limits of geometric objects with affine transition morphisms. The most complete and updated form of this theory can be found in [Ryd1], as well as the history of the subject and references to other sources. Let us recall briefly the main results for algebraic spaces. *Classical approximation* studies filtered projective families of spaces $\{X_\alpha\}$ with affine transition morphisms. It was developed for schemes in [EGA, IV₃, §8], and the case of general spaces (and even stacks) follows rather easily, see [Ryd1, Appendix B]. In particular: (a) there exists a limit $X = \lim_\alpha X_\alpha$, $|X| \xrightarrow{\sim} \lim_\alpha |X_\alpha|$, and the projections $X \rightarrow X_\alpha$ are affine, (b) the category FP/X of finitely presented X -spaces is naturally equivalent to the 2-categorical colimit of the categories FP/X_α , and for almost all geometric properties (e.g. projectivity, smoothness, etc.) a morphism $f: Y \rightarrow Z$ in FP/X satisfies **P** if and only if for sufficiently large α its approximations $f_\alpha: Y_\alpha \rightarrow Z_\alpha$ in FP/X_α satisfy **P**, (c) if $\{X_\alpha\}$ is an S -family and $Y \rightarrow S$ is finitely presented then $\mathrm{colim}_\alpha \mathrm{Mor}_S(X_\alpha, Y) \xrightarrow{\sim} \mathrm{Mor}_S(X, Y)$.

More subtle approximation results are related to filtered families that are subject to certain finite presentation or finite type conditions. *Approximation of modules*, see [Ryd1, Th. A]: any finitely generated module is an epimorphic image of a finitely presented one, and any quasi-coherent \mathcal{O}_X -module \mathcal{L} is the filtered colimit of both: the family of finitely generated submodules, and the family of finitely presented \mathcal{O}_X -modules with a morphism to \mathcal{L} . The same claims hold for quasi-coherent \mathcal{O}_X -algebras. *Approximation of morphisms*, see [Ryd1, Th. D]: any finite type morphism $Y \rightarrow X$ factors as a closed immersion $Y \rightarrow \overline{Y}$ followed by a finitely presented morphism $\overline{Y} \rightarrow X$. Any morphism of algebraic spaces $f: X \rightarrow S$ is a filtered limit of finitely presented morphisms $f: X_\alpha \rightarrow S$ with affine transition morphisms $X_\alpha \rightarrow X_\beta$, and for a list of properties **P**₀ stable under composition and including affine morphisms (e.g. affine, quasi-affine or separated) f satisfies **P**₀ if and only if so do f_α for all α large enough. *Approximation of properties*, see [Ryd1, Th. C]: if all X_α are finitely presented over S then $X \rightarrow S$ satisfies **P**₀ if and only if $X_\alpha \rightarrow S$ do so for all α large enough.

2.2. Generizing sets.

2.2.1. Specialization relation. Given an algebraic space X we denote by \succeq and \preceq the generalization and the specialization relations on $|X|$, and by \succ and \prec the corresponding strict relations. The following notation will be useful: $X_{\succeq y} := \{x \in X \mid x \succeq y\}$, $X_{\succ y} := \{x \in X \mid x \succ y\}$, $X_{\preceq y} := \{x \in X \mid x \preceq y\}$, $X_{\prec y} := \{x \in X \mid x \prec y\}$.

$y\}$. In particular, $X_{\leq y}$ is the closure of y , and if X is a scheme then $X_{\geq y}$ is nothing but the local scheme X_y .

2.2.2. S -topology and generizing sets. The specialization relation induces a topology, which we, following Rydh, call S -topology. Its open (resp. closed) sets are the sets closed under generalization (resp. specialization). For shortness, we call S -open sets *generizing*. Obviously, the family of such sets is closed under arbitrary unions and intersections, and a set is generizing if and only if it is an intersection of Zariski open sets. For any morphism of algebraic spaces $f: Y \rightarrow X$ the underlying continuous map $|f|: |Y| \rightarrow |X|$ preserves the specialization relation, hence is continuous in the S -topology. Thus, the preimage of a generizing set is generizing. In opposite direction, any open or flat morphism f is S -open (or *generizing*) by [LMB, 5.7.1 and 5.8]. In particular, if f is flat then $f(Y_{\geq y}) = X_{\geq f(y)}$ for any $y \in Y$, and the image of a generizing set is generizing.

2.2.3. Zariski trees. Let T be a qcqs schematic topological space (e.g. $T = |X|$ for a qcqs algebraic space X). We say that T is a *Zariski tree* if T is connected and for any point $t \in T$ the set $T_{\geq t}$ is totally ordered by specialization. In particular, if $|X|$ is a Zariski tree then X is irreducible. To justify our terminology, we can view T as a sort of tree directed by the specialization. Its root is the generic point and its leaves are the closed points. *Zariski forest* is a disjoint union of finitely many Zariski trees. A basic example of a Zariski tree is an irreducible algebraic curve or (as we will later see) an irreducible Prüfer scheme.

2.2.4. Zariski chains. By *Zariski chain* we mean a Zariski tree T such that the set of specializations of any point is totally ordered. Since we assume that T is quasi-compact this simply means that T has a unique closed point. One easily sees that any generizing subset $S \subseteq T$ is of the form $T_{\geq t}$ or $T_{> t}$ and S is open (resp. quasi-compact) if and only if it is of the form $T_{> t}$ or coincides with T (resp. is of the form $T_{\geq t}$). In particular, $T_{\geq t}$ is open if and only if t is closed or possesses an immediate specialization. A typical example of a Zariski chain is $\text{Spec}(R)$ for a valuation ring R , because all (including non-prime) ideals of R are totally ordered by inclusion.

Lemma 2.2.1. *Let $\phi: Y \rightarrow X$ be an essentially étale morphism, and assume that $X = \text{Spec}(R)$ for a valuation ring R and $Y = \text{Spec}(A)$ for a local ring A . Then A is a valuation ring, and the induced map $|\phi|: |Y| \rightarrow |X|$ is a topological embedding with generizing image. In particular, $|\phi|$ is a homeomorphism if and only if its image contains the closed point of X .*

Proof. Note that Y is normal because it is essentially étale over the normal scheme X . It follows from locality of A that Y is irreducible. By Zariski main theorem, A is a localization of a finite R -algebra, and using that A is integrally closed in $k(Y)$ we obtain that A is a localization of the integral closure of R in $k(Y)$. Thus, A is a valuation ring by [Bou, Ch.VI, §7, Prop. 1 and 6] and Y is a Zariski chain. Since $|\phi|$ is a generizing map (see §2.2.2), it remains to show that it is injective. Note that ϕ has discrete fibers, since it is a localization of a finite morphism. However, the only discrete subsets in a Zariski chain are points. \square

2.3. Modifications and quasi-modifications.

2.3.1. U -admissibility. Assume that X is an algebraic space and $U \subseteq |X|$ is a generizing subset. We say that U is *schematically dense* if any open subspace containing U is so (recall that a subspace is called schematically dense if it is not contained in any proper closed subspace of X). If $U \subset X$ is schematically dense, and $f: X' \rightarrow X$ is a morphism then f is called *U -admissible* if $f^{-1}(U)$ is schematically dense in X' .

2.3.2. U -modifications. Let $U \subset |X|$ be a schematically dense generizing subset. By *U -modification* (resp. *U -quasi-modification*) we mean a proper (resp. separated finite type) U -admissible morphism $f: X' \rightarrow X$ such that there exists an open subspace $V \subseteq X$ containing U for which $f^{-1}(V) \rightarrow V$ is an isomorphism (resp. an open immersion). In general, by *modification* (resp. *quasi-modification*) we mean a morphism $f: X' \rightarrow X$ which is a U -modification (resp. U -quasi-modification) for some choice of an open schematically dense subspace $U \hookrightarrow X$.

Remark 2.3.1. If X' and X'' are two U -modifications of X then there exists at most one X -morphism $f: X'' \rightarrow X'$ because U is schematically dense in X'' and X' is X -separated. In this case we say that X'' *dominates* X' . The set of all U -modifications of X ordered by domination is filtered because U -modifications X', X'' are dominated by the schematic image of U in $X' \times_X X''$. The same facts hold true for the family of modifications of X .

2.3.3. Blow ups. The *blow up* of X along an ideal $\mathcal{I} \subset \mathcal{O}_X$ is defined as $\mathrm{Bl}_{\mathcal{I}}(X) := \mathbf{Proj}(\oplus_{n=0}^{\infty} \mathcal{I}^n)$. If X is reduced and the closed immersion $Z = \mathbf{Spec}(\mathcal{O}_X/\mathcal{I}) \hookrightarrow X$ is finitely presented and nowhere dense then the blow up $f: \mathrm{Bl}_{\mathcal{I}}(X) \rightarrow X$ is a modification. If, in addition, \mathcal{I} is *U -trivial* in the sense that $|Z|$ is disjoint from U then f is a U -modification.

2.4. Pro-open immersions. In this section we introduce and study pro-open immersions. This is the only section where we do not impose the quasi-compactness assumption almost everywhere. In the quasi-compact case, the families of pro-open immersion and flat monomorphisms coincide, but the former family behaves better in general. In the category of schemes, quasi-compact flat monomorphisms were studied by Raynaud in [Ray]. The general case can be reduced to Raynaud's work as soon as one shows that pro-open immersions are schematic. Surprisingly, this is not easy, and two proofs that we know heavily rely on the approximation theory. The proof we present below is longer, but it is based on Proposition 2.4.6, which seems to be of independent interest.

2.4.1. Definition and basic properties. Let X be an algebraic space, and $U \subseteq |X|$ be a generizing subset. Consider the functor h_U that associates to an algebraic space Y the set of morphisms $Y \rightarrow X$ having set-theoretical image in U . If h_U is representable by an algebraic space \mathcal{U} then we say that U is a *pro-open subspace* and (by a slight abuse of language) identify U with \mathcal{U} . This makes sense because the natural map $|\mathcal{U}| \rightarrow U$ is clearly a bijection. We will show below that $|\mathcal{U}| \rightarrow U$ is a homeomorphism; thus, \mathcal{U} is the topological space U equipped with an additional structure defined uniquely by U and X . Any morphism isomorphic to $\mathcal{U} \rightarrow X$ as above is called a *pro-open immersion*. A typical example of a pro-open immersion is the localization morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \rightarrow X$ for a scheme X and a point $x \in X$. Let us list some simple properties of pro-open immersions:

Proposition 2.4.1. (i) A generizing subset U in an algebraic space X is a pro-open subspace if and only if the limit of the family of all open subspaces $U_\alpha \hookrightarrow X$ with $U \subset |U_\alpha|$ is representable. In this case, $\mathcal{U} = \lim_\alpha U_\alpha$ is the pro-open subspace corresponding to U .

(ii) Any pro-open immersion $i: \mathcal{U} \rightarrow X$ is a flat monomorphism.

(iii) If $i: \mathcal{U} \rightarrow X$ is a pro-open immersion, and $U \subset |X|$ is its image then $|U| \rightarrow U$ is a homeomorphism in the S -topology.

(iv) The class of pro-open immersions is closed under compositions and base changes.

(v) If $i: \mathcal{U} \rightarrow X$ and $j: \mathcal{V} \rightarrow X$ are pro-open immersions and $i(|U|) \subseteq j(|V|)$ then the natural morphism $\mathcal{U} \rightarrow \mathcal{V}$ is a pro-open immersion.

(vi) The class of pro-open immersions is closed under fpqc descent. Namely, if $i: U \rightarrow X$ is a morphism of algebraic spaces and $i \times_X X'$ is a pro-open immersion for a flat quasi-compact morphism $X' \rightarrow X$ then i itself is a pro-open immersion.

Proof. Note that assertion (i), the property of being a monomorphism, and the stability under base changes follow from the universal property in the definition. For example, we have an isomorphism of functors $h_U \xrightarrow{\sim} \lim_\alpha h_{U_\alpha}$, regardless to their representability. Since U_α represents h_{U_α} , this implies (i). To prove (ii), pick a presentation $g: X'' \rightarrow X$. Set $X' := \sqcup_{x \in g^{-1}(f(Y))} \text{Spec}(\mathcal{O}_{X'', x})$, then $X' \rightarrow X$ is faithfully flat. The base change $f': \mathcal{U} \times_X X' \rightarrow X'$ is a surjective pro-open immersion, hence an isomorphism. Thus, $\mathcal{U} \rightarrow X$ is flat by faithfully flat descent.

The assertion of (iii) holds for any flat injective morphism, hence (ii) implies (iii). Assertion (v), and the stability under compositions follow from (iii), so it remains to prove (vi). Set $U' := U \times_X X'$, $X'' := X' \times_X X'$ and $U'' := U \times_X X''$. By (iii), $U'' \rightarrow X''$ is a pro-open immersion. For any morphism $f: Y \rightarrow X$ with $f(|Y|) \subseteq U$, its base changes $f': Y' \rightarrow X'$ and $f'': Y'' \rightarrow X''$ land in $|U'|$ and $|U''|$ respectively. Hence f' and f'' induce morphisms $g': Y' \rightarrow U'$ and $g'': Y'' \rightarrow U''$, and these morphisms descent to an X -morphism $g: Y \rightarrow U$ by fpqc descent. Therefore, i is a pro-open immersion. \square

In addition, we will later see that any pro-open immersion f is schematic and a topological embedding.

2.4.2. Relation to flat monomorphisms. It is well known that open immersions can be characterized as locally finitely presented flat monomorphisms. Quasi-compact pro-open immersions admit similar characterization:

Theorem 2.4.2. Assume that $f: Y \rightarrow X$ is a quasi-compact morphism of algebraic spaces. Then f is a pro-open immersion if and only if f is a flat monomorphism.

Proof. The direct implication holds even without the quasi-compactness assumption by Proposition 2.4.1(ii). Assume that f is a flat monomorphism. Recall that $U = f(Y)$ is generizing by flatness of f . The diagonal $Y \rightarrow Y \times_X Y$ is an isomorphism, hence so is any projection $Y \times_X Y \rightarrow Y$. The latter is the base change of f with respect to itself, so if f is surjective then it is an isomorphism by fpqc descent. In general, given a morphism $h: Z \rightarrow X$ with image in U , the base change $Z \times_X Y \rightarrow Z$ is a quasi-compact surjective flat monomorphism, hence an isomorphism. Thus, h factors uniquely through Y . \square

The following example shows that the quasi-compactness assumption is essential. Note that for open immersions one can avoid it thanks to the l.f.p. assumption.

Example 2.4.3. Let C be an integral curve with infinitely many closed points. Consider the curve C' obtained by gluing all localizations $C_x = \operatorname{Spec}(\mathcal{O}_{C,x})$ along their generic points. Then the natural morphism $C' \rightarrow C$ is a bijective flat monomorphism, which is not quasi-compact. Actually C' is obtained from C by replacing the topology of C with the S -topology, and “preserving” the structure sheaf.

2.4.3. Quasi-compact pro-open immersions. Quasi-compactness assumption makes it possible to strengthen some basic assertions about pro-open immersions. Two such basic properties are the faithfully flat descent (without the fpqc assumption), and the locality on the source:

Proposition 2.4.4. (i) *Let $i: Y \rightarrow X$ be a quasi-compact morphism, and $X' \rightarrow X$ be a flat surjective morphism. If $i \times_X X'$ is a pro-open immersion then so is i .*

(ii) *Let $f: Y \rightarrow X$ be a separated quasi-compact morphism. If Y can be covered by pro-open subspaces Y_i such that the compositions $f_i: Y_i \rightarrow X$ are pro-open immersions then f is a pro-open immersion.*

Proof. Clearly, flatness is preserved by arbitrary faithfully flat descent. Moreover, the same is true for being a monomorphism, and for equality of morphisms; e.g. $p_{1,2}: Z \rightarrow Y$ are equal if and only if $p_{1,2} \times_X X'$ are equal. Therefore, (i) follows from Theorem 2.4.2. Now, let f be as in (ii). Set $U := f(Y) = \cup f_i(Y_i)$, and pick a presentation $g: X' \rightarrow X$. Then $U \subseteq X$ is generizing, and for any $x \in g^{-1}(U)$ there exists i such that the morphism $f_{i,x} := f_i \times_X X'_x$ is a surjective pro-open immersion, hence an isomorphism, where X'_x denotes the local scheme $\operatorname{Spec}(\mathcal{O}_{X',x})$. Thus, $Y \times_X X'_x \rightarrow X'_x$ is an isomorphism, since f is separated and $Y_i \rightarrow Y$ is a pro-open immersion. Then $\coprod_{x \in X'} Y \times_X X'_x \rightarrow \coprod_{x \in X'} X'_x$ is a flat monomorphism, and it follows by descent that f is so, hence a pro-open immersion. \square

2.4.4. Factorization of pro-open immersions. It might look plausible to expect that if a pro-open immersion $i: U \rightarrow X$ factors through a finite type morphism $f: Y \rightarrow X$ so that the morphism $j: U \rightarrow Y$ is schematically dominant then there exists an open neighborhood of $j(|U|)$ such that the restriction of f to it is an open immersion. Surprisingly, this is wrong, and some additional assumptions are needed.

Example 2.4.5. (i) The following example was communicated to the first author by B. Conrad: Let X be a zero dimensional qcqs scheme with a non-discrete point $x \in X$. For example, one can consider an infinite product of fields $A = \prod_{i \in I} k_i$. Then $X = \operatorname{Spec}(A)$ is a compact topological space naturally homeomorphic to the Stone-Ćech compactification \widehat{I} of the set I provided with the discrete topology. Clearly, X contains many non-discrete points – these are precisely the points of $\widehat{I} \setminus I$. Note that $\mathcal{O}_{X,x} = k(x)$, hence the closed immersion $i: x \hookrightarrow X$ is a local-Ćtale morphism of finite type. On the other hand, i is not Ćtale because it is not finitely presented. The latter was, actually, the goal of Conrad’s example. In addition, i is flat and of finite type but not of finite presentation. Note for the sake of comparison, that if X is integral then any flat finite type morphism $Y \rightarrow X$ is of finite presentation by [RG, 3.4.7]. Returning to our situation, note that $x \rightarrow X$ is a pro-open immersion of finite type which is not an open immersion. In particular, the naive expectation in the beginning of §2.4.4 is wrong when $U = Y = x$.

(ii) Another example, that goes back to J.-P. Olivier [O], was suggested to us by Rydh. It is known that there exists an affine bijective monomorphism

$X = \operatorname{Spec}(A) \rightarrow \operatorname{Spec}(\mathbf{Z})$ such that X is universally flat (i.e. any A -module is flat), and the topology on X is the constructible topology of $\operatorname{Spec}(\mathbf{Z})$. In particular, X is a zero dimensional qcqs scheme with a non-discrete point $\operatorname{Spec}(\mathbf{Q})$ and discrete points $\operatorname{Spec}(\mathbf{F}_p)$. Moreover, A can be described explicitly as follows: $A = \mathbf{Z}[T_2, T_3, T_5, \dots]/I$, where I is the ideal generated by $(pT_p - 1)p$ and $(pT_p - 1)T_p$ for all prime $p \in \mathbf{Z}$. Set $U = Y := \operatorname{Spec}(\mathbf{Q})$. Then $U \rightarrow X$ is a pro-open immersion, $Y \rightarrow X$ is of finite type, but not an open immersion.

A standard way to strengthen the assumptions on U, Y , and X is to require that $f: Y \rightarrow X$ is of finite presentation. This indeed makes the claim correct, but we will have to work with more general morphisms of finite type. Fortunately, there is another assumption that eliminates bad examples as above and corrects the claim.

Proposition 2.4.6. *Assume that U, Y, X are qcqs algebraic spaces, $j: U \rightarrow Y$ is a schematically dominant morphism, $f: Y \rightarrow X$ is a separated morphism such that the composition $i: U \rightarrow X$ is a pro-open immersion, and one of the following conditions is satisfied:*

- (i) *f is of finite presentation,*
- (ii) *f is of finite type and i is schematically dominant.*

Then, for a sufficiently small open neighborhood $V \subset X$ of $i(U)$, the restriction $f \times_X V$ is an isomorphism. In particular, any pro-open immersion f that satisfies (i) or (ii) is an open immersion.

Proof. Step 0. *General setup.* If $X' \rightarrow X$ is a presentation and the assertion of the proposition holds for $f' = f \times_X X'$ and $j' = j \times_X X'$ then it also holds for f and j by descent. Thus, we can restrict the consideration to the case when X is an affine scheme. In the sequel we will use the following notation: $u \rightarrow U$ is a Zariski point, $y = j(u)$, $x = i(u)$, $X_x = \operatorname{Spec}(\mathcal{O}_{X,x})$, $Y_x = X_x \times_X Y$ and $U_x = X_x \times_X U$. Note that X_x is the limit of affine neighborhoods, so we may use the classical approximation theory for this limit. Also, by approximation of morphisms we can realize Y as a closed subspace in an algebraic space \overline{Z} of finite presentation over X , and we fix such a closed immersion $Y \hookrightarrow \overline{Z}$.

Step 1. *For any choice of $u \in U$, the morphism $Y_x \rightarrow X_x$ is an isomorphism.* Note that $U_x \rightarrow X_x$ is an isomorphism because it is a surjective pro-open immersion by Proposition 2.4.1. To complete the step we should prove the following lemma.

Lemma 2.4.7. *Let $\pi: Y \rightarrow X$ be a separated morphism of algebraic spaces, and $\iota: X \rightarrow Y$ be its section. If ι is schematically dominant then π is an isomorphism.*

Proof. Since $\pi\iota = id$, it is sufficient to show that $\iota\pi = id$, or, equivalently, that $id \times \iota\pi: Y \rightarrow Y \times Y$ factors through the diagonal embedding $\Delta_Y: Y \rightarrow Y \times Y$. Plainly, $id \times \iota\pi$ factors through $Y \times_X Y$, since ι is a section. Consider the corresponding diagram of solid arrows with cartesian squares:

$$\begin{array}{ccccc}
 Y \times_{Y \times_X Y} Y & \xrightarrow{\quad} & Y & \xrightarrow{id} & Y \\
 \downarrow \Delta' & \nearrow \iota & \downarrow \Delta_{Y/X} & & \downarrow \Delta_Y \\
 & X & & & \\
 & \nwarrow \iota & & & \\
 Y & \xrightarrow{id \times_X \iota\pi} & Y \times_X Y & \longrightarrow & Y \times Y
 \end{array}$$

The dotted arrow exists, since $\iota\pi\iota = \iota$. Since π is separated, $\Delta_{Y/X}$ is a closed immersion, hence Δ' is so; and since ι is schematically dominant, this implies that $Y \times_{Y \times_X Y} Y = Y$ and $\Delta' = id$. Thus, $id \times \iota\pi$ factors through Δ_Y . \square

If f is finitely presented then it is easy to complete the proof (see below). The main effort will be to approximate Y with a finitely presented subspace $Z \hookrightarrow \overline{Z}$ such that $Y \hookrightarrow Z$ and Z satisfies the assertion of Step 1. This will be achieved in Steps 2 and 3. In these steps we only consider case (ii).

Step 2. *Fix $u \in U$. Then one can find a finitely presented closed immersion $T = T(u) \hookrightarrow \overline{Z}$ with $Y \hookrightarrow T$ such that $X_x \times_X T \rightarrow X_x$ is an isomorphism.* Let $\mathcal{I} \subset \mathcal{O}_{\overline{Z}}$ be the ideal that defines Y , and let \mathcal{I}_x be its restriction (or pullback) to $\overline{Z}_x := X_x \times_X \overline{Z}$. The closed subspace $Y_x \hookrightarrow \overline{Z}_x$ is finitely presented over X_x , hence given by a finitely generated $\mathcal{O}_{\overline{Z}_x}$ -ideal $\mathcal{J}_x \supseteq \mathcal{I}_x$. We claim that \mathcal{J}_x can be extended to a finitely generated \mathcal{O}_Z -ideal $\mathcal{J} \supseteq \mathcal{I}$: first extend it over a neighborhood of x using approximation of modules, then extend it from the corresponding open subspace by [Ryd1, Th. A]. Then \mathcal{J} defines a finitely presented closed subspace $T \hookrightarrow \overline{Z}$ as required.

Step 3. *There exists finitely presented closed immersion $Z \hookrightarrow \overline{Z}$ with $Y \hookrightarrow Z$ such that $X_x \times_X Z \rightarrow X_x$ is an isomorphism for any choice of $u \in U$.* For any choice of $u \in U$ find a finitely presented $Y \hookrightarrow T(u)$ as in Step 2. Since $h(u): T(u) \rightarrow X$ is finitely presented and $h(u) \times_X X_x$ is an isomorphism, classical approximation implies that there exists an open affine neighborhood $x \in W(u)$ such that $h(u) \times_X W(u)$ is an isomorphism. Since U is quasi-compact, there exist points u_1, \dots, u_n such that $U \subseteq \bigcup_{i=1}^n W(u_i)$. Then we can take Z to be the scheme-theoretic intersection of $T(u_1), \dots, T(u_n)$. Indeed, it follows from Lemma 2.4.7 that $X_x \times_X Z \xrightarrow{\sim} X_x$ for any choice of u .

The rest is easy. The same approximation argument as above shows that $T \rightarrow X$ is a local isomorphism at all points of the image of U and hence there exists an open quasi-compact neighborhood $V \hookrightarrow X$ of $|U|$ such that $Z \times_X V \rightarrow V$ is an isomorphism. This is enough to establish case (i) since in this case $Y = Z$. In case (ii), after replacing X, Y and Z with V and its base changes, we may assume that $Z \rightarrow X$ is an isomorphism. Thus, $U \rightarrow Z \xrightarrow{\sim} X$ is schematically dominant, and hence so is $Y \hookrightarrow Z$. But then $Y \xrightarrow{\sim} Z$, and we are done. \square

2.4.5. Pro-open immersions and approximation. In Proposition 2.4.1(i), we have claimed that any pro-open subspace $U \hookrightarrow X$ is the projective limit of its open neighborhoods U_α . This does not allow to apply directly the approximation machinery because it is unclear whether there exists a cofinal subfamily such that all its transition morphisms are affine. Fortunately, there is a less direct way based on Proposition 2.4.6. Indeed, it immediately implies the following result, where by a strict U -modification $X' \rightarrow X$ we mean a U -quasi-modification whose restriction onto a neighborhood of U in X is an isomorphism.

Corollary 2.4.8. *Assume that $i: U \rightarrow X$ is a schematically dominant pro-open immersion of qcqs algebraic spaces and $\{U_\alpha\}_{\alpha \in A}$ is the family of open subspaces of X with $|U| \subset |U_\alpha|$. Then U_α are cofinal among the family $\{Y_\beta\}_{\beta \in B}$ of all strict U -quasi-modifications of X .*

On the other hand, by approximation of morphisms, we can find a strict U -quasi-modification $Y \rightarrow X$ such that the morphism $U \rightarrow Y$ is affine, and then U is

isomorphic to the limit of all strict U -quasi-modifications of X that dominate Y and are affine over it. In particular, the family $\{Y_\beta\}_{\beta \in B}$ contains a cofinal subfamily on which classical approximation works. Hence it also applies to the whole family $\{Y_\beta\}_{\beta \in B}$ and to its subfamily $\{U_\alpha\}_{\alpha \in A}$.

Corollary 2.4.9. *Assume that $i: U \rightarrow X$ is a schematically dominant pro-open immersion of qcqs algebraic spaces and $\{U_\alpha\}_{\alpha \in A}$ is the family of open subspaces of X with $|U| \subset |U_\alpha|$. Then all results of the classical approximation theory mentioned in §2.1.4 hold true for the limit $U = \lim_{\alpha \in A} U_\alpha$.*

2.4.6. *Pro-open immersions on the level of locally ringed spaces.* Now, we can complete our list of basic properties of pro-open immersions.

Proposition 2.4.10. (i) *Any pro-open immersion $i: U \hookrightarrow X$ is a schematic morphism that induces a topological embedding $|U| \hookrightarrow |X|$.*

(ii) *Any quasi-compact pro-open immersion is quasi-affine.*

(iii) *A generizing subset U in a scheme X is a pro-open subscheme if and only if the locally ringed space $\mathcal{U} = (U, \mathcal{O}_X|_U)$ is a scheme, and then \mathcal{U} is the pro-open subscheme corresponding to U .*

Proof. By descent we can assume that X is an affine scheme. Also, claims (i) and (iii) are Zariski-local on the source, so we can always assume that the space U is quasi-compact. It now suffices to prove that U is actually a scheme because in this case Raynaud proved our claim in [Ray, §1]. Let X' be the schematic image of U in X . Then $U \xrightarrow{\sim} U \times_X X'$ and hence $U \rightarrow X'$ is a pro-open immersion. Replacing X with X' we can assume that i is schematically dominant. Then, as we showed in §2.4.5, we can factor i as $U \rightarrow U_\alpha \rightarrow Y \rightarrow X$ where $U \rightarrow Y$ is affine. Since U_α is X -separated, hence Y -separated, we obtain that $U \rightarrow U_\alpha$ is affine and hence $U \rightarrow X$ is quasi-affine. \square

2.4.7. *Affine pro-open immersions.* Using Proposition 2.4.10(ii) one can easily prove the following description of affine pro-open immersions.

Proposition 2.4.11. *Assume that X is an algebraic space with a flat quasi-compact presentation $X' \rightarrow X$ and a generizing subset $U \subseteq |X|$. Then U is a pro-open subspace with an affine pro-open immersion $U \rightarrow X$ if and only if for any point $x \in X'$ the preimage of U in $\mathrm{Spec}(\mathcal{O}_{X',x})$ is an affine pro-open subscheme.*

Proof. We should only establish the inverse implication. By Proposition 2.4.1(vi) it suffices to establish the case when $X = X'$, in particular, X is a scheme. Then this follows from [Ray, Prop. 2.4 B(iv)]. \square

2.4.8. *Localization of algebraic spaces.* Let X be an algebraic space with a point x . If $X_{\geq x}$ is a pro-open subspace then we call it the *localization* of X at x ; sometimes it will be denoted X_x . One of technical obstacles in the proof of Nagata compactification in [TT] is that localization does not exist in general. A counter-example can be obtained already for Hironaka's smooth proper threefold, see [Har, Appendix B, Example 3.4.2]. This was discovered in our discussion with Rydh, and a detailed presentation will be given elsewhere.

3. FERRAND'S PUSHOUTS

The aim of §3 is to study pushouts of algebraic spaces $Y \sqcup_T Z$ in the special case when $T \rightarrow Y$ is an affine morphism, $T \rightarrow Z$ is a closed immersion, and T, Y, Z are

qcqs. The affine case was studied extensively by D. Ferrand, so we will call such pushouts Ferrand's (see §3.2.1).

Pushouts discover very subtle and unstable behavior when working with geometric categories, including those of affine schemes, schemes, algebraic spaces, and stacks. This is not so surprising since these operations include the quotient by a group action or an equivalence relation. In particular, it often happens that two geometric categories $\mathfrak{C} \subset \mathfrak{C}'$ possess different pushouts $X = Y \sqcup_T^{\mathfrak{C}} Z$ and $X' = Y \sqcup_T^{\mathfrak{C}'} Z$. In such case the natural \mathfrak{C}' -morphism $f: X' \rightarrow X$, that represents the morphisms from X' to objects of \mathfrak{C} , is not an isomorphism, and one can view X' as a finer or a more informative pushout. The classical example is when X' is a quotient stack, and X is the coarse quotient (or coarse moduli space of X'). Also, see Remarks 3.1.2 and 5.1.3 for analogous examples with other categories. We will see that Ferrand's pushouts discover a much better behavior.

3.1. The affine case.

3.1.1. Pushout data. By *pushout datum* we mean any pair of morphisms $f: T \rightarrow Y$ and $g: T \rightarrow Z$. If not said to the contrary, X , Y and Z are assumed to be algebraic spaces. Such datum will be denoted $\mathcal{P} = (T; Y, Z)$, $\mathcal{P}' = (T'; Y', Z')$, etc. Sometimes we will view an algebraic space U as the trivial datum $(U; U, U)$. A *morphism* $\mathcal{P}' \rightarrow \mathcal{P}$ is a triple of morphisms $\alpha: T' \rightarrow T$, $\beta: Y' \rightarrow Y$ and $\gamma: Z' \rightarrow Z$ such that α is the base change of both β and γ , and instead of writing $\mathcal{P} \rightarrow (U; U, U)$ we will usually write $\mathcal{P} \rightarrow U$. Similarly, we set $\mathcal{O}_{\mathcal{P}} := (\mathcal{O}_T; \mathcal{O}_Y, \mathcal{O}_Z)$, and by an $\mathcal{O}_{\mathcal{P}}$ -module we mean a triple of modules $\mathcal{M} = (\mathcal{M}_T; \mathcal{M}_Y, \mathcal{M}_Z)$ with isomorphisms $f^*(\mathcal{M}_Y) \xrightarrow{\sim} \mathcal{M}_T$ and $g^*(\mathcal{M}_Z) \xrightarrow{\sim} \mathcal{M}_T$. Unless explicitly said to the contrary, we say that a morphism of pushout data (resp. a pushout datum) possesses a certain property if all its components do. In particular, if \mathfrak{C} is the category of algebraic spaces then we define affine pushout data, and étale or affine morphisms via this rule. Similarly, an *affine presentation* of a pushout datum is a surjective étale morphism to it from an affine pushout datum.

3.1.2. General S -affine pushouts. It is well known that pushouts exist in the category of affine schemes. We will need in [TT] the following relative version of this observation. By an *S -affine pushout* we mean pushout in the category of S -affine algebraic spaces. Such pushout will be denoted $Y \sqcup_T^{S\text{-aff}} Z$.

Lemma 3.1.1. *Let S be an algebraic space and let $\mathcal{P} = (T; Y, Z)$ be an S -affine pushout datum with $Y = \mathbf{Spec}(\mathcal{B})$, $Z = \mathbf{Spec}(\mathcal{C})$, and $T = \mathbf{Spec}(\mathcal{K})$. Then there exists a pushout $X = Y \sqcup_T^{S\text{-aff}} Z$ in the category of S -affine spaces given by $X = \mathbf{Spec}(\mathcal{A})$, where $\mathcal{A} = \mathcal{B} \times_{\mathcal{K}} \mathcal{C}$ in the category of \mathcal{O}_S -algebras. Moreover, if $S' \rightarrow S$ is a flat morphism, $\mathcal{P}' = \mathcal{P} \times_S S'$, and $X' = \mathbf{Spec}(\mathcal{A}')$ is the S' -pushout of \mathcal{P}' then $\mathcal{A}' = \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$.*

Proof. The first claim is obvious, since the category of S -affine spaces is anti-equivalent to the category of quasi-coherent \mathcal{O}_S -algebras. For the moreover part, note that the sequence $0 \rightarrow \mathcal{K} \rightarrow \mathcal{B} \oplus \mathcal{C} \rightarrow \mathcal{A} \rightarrow 0$ is exact, hence its pull-back $0 \rightarrow \mathcal{K}' \rightarrow \mathcal{B}' \oplus \mathcal{C}' \rightarrow \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \rightarrow 0$ is exact. Thus, $\mathcal{A}' = \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$. \square

Remark 3.1.2. Affine pushouts are often rather meaningless. If a pushout in the category of schemes (or algebraic spaces) exists then affine pushout is its affine hull. For example, \mathbf{P}_k^1 is the pushout of $\mathbf{Spec}(k[T])$ and $\mathbf{Spec}(k[T^{-1}])$ along

$\mathrm{Spec}(k[T, T^{-1}])$, but the affine pushout is $\mathrm{Spec}(k)$. This shows that even when the morphisms $T \rightarrow Y$ and $T \rightarrow Z$ are open immersions the affine pushout can be bad.

3.1.3. S -affine Ferrand's pushouts. It turns out that S -affine pushouts behave much better when one of the morphisms is a closed immersion. We assume in the sequel that $T \rightarrow Z$ is a closed immersion and say that $\mathcal{P} = (T; Y, Z)$ is *Ferrand's S -affine pushout datum* and $X = Y \sqcup_T^{S^{\mathrm{aff}}} Z$ is *Ferrand's S -affine pushout* because the absolute case (when $S = \mathrm{Spec}(\mathbf{Z})$ and A, B, C, K are rings) was studied by Ferrand in [Fer]. Here are the main good properties of such pushouts:

Proposition 3.1.3. *Any Ferrand's S -affine pushout $X = Y \sqcup_T^{S^{\mathrm{aff}}} Z$ satisfies the following properties:*

- (i) *the topological pushout $|Y| \sqcup_{|T|} |Z|$ is naturally homeomorphic to $|X|$,*
- (ii) *pulling back an \mathcal{O}_X -module to $(T; Y, Z)$ induces an equivalence between the category of flat quasi-coherent \mathcal{O}_X -modules and the category of flat quasi-coherent $\mathcal{O}_{\mathcal{P}}$ -modules,*
- (iii) *$T \rightrightarrows Y \times_X Z$ and $Y \rightarrow X$ is a closed immersion.*

Proof. In the absolute case the claim was proved by Ferrand, see [Fer, Th. 2.2(iv) and Scholie 4.3]. In particular, this establishes the case when S is affine. In general, pick an étale presentation $S_1 \rightrightarrows S_0$ of S with affine S_0 . First, assume that S is separated and so S_1 is affine too. Then $X_i := X \times_S S_i$ is the affine pushout of $\mathcal{P}_i = \mathcal{P} \times_S S_i$ by Lemma 3.1.1. Both pushouts are absolute Ferrand's pushouts; hence satisfy (i), (ii), and (iii). It then follows by descent that the original S -affine pushout satisfies these properties. Now let us assume that the algebraic space S is arbitrary. Although S_1 does not have to be affine in this case, it is necessarily separated. So, the S_1 -affine pushout $X_1 = Y_1 \sqcup_{T_1}^{S_1^{\mathrm{aff}}} Z_1$ satisfies (i), (ii) and (iii) thanks to the first case considered above. Thus, the same descent argument applies, and the Proposition follows. \square

Remark 3.1.4. (i) The first assertion of Proposition 3.1.3 can be interpreted as an equivalence of Zariski sites $\phi_{\mathrm{Zar}}: \mathrm{Zar}/X \xrightarrow{\sim} \mathrm{Zar}/\mathcal{P}$, where open immersions in the category of pushout data are defined by our conventions, and ϕ_{Zar} is induced by the base change with respect to the morphism $\mathcal{P} \rightarrow X$.

(ii) Similarly, the second assertion of the Lemma implies that the natural functor between the flat affine sites $\phi_{\mathrm{Fl}^{\mathrm{aff}}}: \mathrm{Fl}^{\mathrm{aff}}/X \rightarrow \mathrm{Fl}^{\mathrm{aff}}/\mathcal{P}$ is an equivalence. Moreover, using the fiberwise criterion of étaleness for flat morphisms and the fact that $|X|$ is covered by $|Y|$ and $|Z|$, we also obtain an equivalence of étale affine sites $\phi_{\mathrm{Ét}^{\mathrm{aff}}}: \mathrm{Ét}^{\mathrm{aff}}/X \xrightarrow{\sim} \mathrm{Ét}^{\mathrm{aff}}/\mathcal{P}$.

Here are few more useful properties of Ferrand's pushouts. Part (ii) of this Lemma shows that these pushouts can be computed flat-locally.

Lemma 3.1.5. *Let $S, \mathcal{P} = (T; Y, Z)$, and X be as above. Then the S -affine pushout X satisfies the following two properties:*

- (i) *Affine uniformness: if $X' \rightarrow X$ is a flat affine morphism and $(T'; Y', Z') = \mathcal{P} \times_X X'$ then the natural morphism $Y' \sqcup_{T'}^{S^{\mathrm{aff}}} Z' \rightarrow X'$ is an isomorphism.*
- (ii) *Affine effectivity: assume that $\mathcal{P}_0 \rightarrow \mathcal{P}$ is a flat surjective morphism of affine pushout data, $\mathcal{P}_1 = \mathcal{P}_0 \times_{\mathcal{P}} \mathcal{P}_0$, and $X_i = Y_i \sqcup_{T_i}^{S^{\mathrm{aff}}} Z_i$. Then $X_1 \rightrightarrows X_0 \times_X X_0$, and so $X_1 \rightrightarrows X_0$ is a flat affine presentation of X .*

Proof. The first claim is proved using the same exact sequence as in the proof of Lemma 3.1.1. The second claim follows from the fact that $\phi_{\mathrm{Fl}^{\mathrm{aff}}}$ is an equivalence; hence preserves fibred products (we use here the fact that fibred products in $\mathrm{Fl}^{\mathrm{aff}}/X$ are the usual fibred products of schemes). \square

3.2. The general case.

3.2.1. *The definition.* Pushouts in our default geometric category - the category of algebraic spaces - will be denoted without mentioning the category. We will be only interested in the following class of pushout data, which we call *Ferrand's pushout data*: $\mathcal{P} = (T; Y, Z)$ such that $T \rightarrow Y$ is affine, $T \rightarrow Z$ is a closed immersion, and T, Y, Z are qcqs. If \mathcal{P} possesses a pushout $X = Y \sqcup_T Z$ such that the morphisms $Y \rightarrow X$ and $Z \rightarrow X$ are affine then we say that the datum is *effective* and the pushout is *Ferrand's pushout*. We do not know if all Ferrand's pushout data are effective, but we will establish effectivity in some cases. In particular, we will show that all S -affine Ferrand's pushouts are Ferrand's pushouts in the category of algebraic spaces, which strengthens Ferrand's result saying that affine Ferrand's pushouts are pushouts in the category of schemes. Note also that D. Rydh showed in [Ryd3, §6] that the same is true in the category of stacks with quasi-affine diagonal if one assumes, in addition, that $T \rightarrow Y$ is a flat schematically dominant monomorphism. In particular, we will prove that, at least in the category of algebraic spaces, the assumption on $T \rightarrow Y$ is not needed.

3.2.2. *Affine presentations.* Note that any effective Ferrand's pushout datum \mathcal{P} possesses an affine presentation $\mathcal{P}' \rightarrow \mathcal{P}$. Indeed, since Y, Z and T are affine over $X = Y \sqcup_T Z$, one can take any affine presentation $X' \rightarrow X$ and set $\mathcal{P}' = \mathcal{P} \times_X X'$. Thus, non-existence of affine presentations is an obstacle to effectivity. One of our main results on Ferrand's pushouts will be that this is the only obstacle. Let us describe two cases when affine presentations exist.

Lemma 3.2.1. *Let \mathcal{P} be a Ferrand's pushout datum.*

(i) *Assume that any étale morphism $T' \rightarrow T$ with an affine source is the base change of an étale morphism $Z' \rightarrow Z$ with an affine source; for example, this holds when $|T|$ is discrete. Then \mathcal{P} admits an affine presentation.*

(ii) *Assume that \mathcal{P} admits a flat quasi-affine morphism to a Ferrand's pushout datum \mathcal{P}' which possesses a flat affine presentation $\mathcal{P}'_0 \rightarrow \mathcal{P}'$. Then \mathcal{P} admits a flat affine presentation $\mathcal{P}_0 \rightarrow \mathcal{P}$, which can be chosen to be étale whenever $\mathcal{P} \rightarrow \mathcal{P}'$ and $\mathcal{P}'_0 \rightarrow \mathcal{P}'$ are étale.*

Proof. (i) Take any affine presentation $Y_0 \rightarrow Y$. Then $T_0 \rightarrow T$ is an affine presentation that can be lifted to an étale morphism $Z'_0 \rightarrow Z$ with an affine source. The latter does not have to be a presentation, but since $T \hookrightarrow Z$ is a closed immersion it is easy to fix this. Namely, take any affine presentation $Z''_0 \rightarrow Z \setminus T$ and observe that $Z_0 := Z'_0 \sqcup Z''_0$ is an affine presentation of Z such that $Z_0 \times_Z T \xrightarrow{\sim} T_0$. In particular, $\mathcal{P}_0 = (T_0; Y_0, Z_0)$ is an affine presentation of \mathcal{P} .

Now, let us explain why the condition of (i) holds when $|T|$ is discrete. Clearly, it is enough to assume that $|T|$ is a point, which we can identify with a Zariski point $z \rightarrow Z$. Any algebraic space is generically a scheme, so T is a scheme and $z = T_{\mathrm{red}}$. By [Knu, Th. II.6.4] there exists an affine presentation $Z_0 \rightarrow Z$ which is strictly étale over z , since $z \in Z$ is a closed point. Thus, the closed immersion $T \hookrightarrow Z$ factors through Z_0 . Since T' is discrete, we can use the local description of

étale morphisms [EGA, IV₄, 18.4.6], to lift the étale morphism $T' \rightarrow T$ to an affine étale morphism $Z' \rightarrow Z_0$.

Remark 3.2.2. A similar argument would not apply to schemes and Zariski topology. For example, if Y and Z are schemes, Y is local and T is not contained in an affine subscheme of Z then \mathcal{P} does not admit an affine Zariski presentation $h: \mathcal{P}' \rightarrow \mathcal{P}$ (i.e. h is a disjoint union of open immersions $h_i: \mathcal{P}'_i \hookrightarrow \mathcal{P}$). In such case, the algebraic space pushout exists (as we will show later) but is not a scheme.

(ii) It is enough to find an appropriate presentation of $\mathcal{P}'_0 \times_{\mathcal{P}'} \mathcal{P}$ because it is also a presentation of \mathcal{P} . Therefore, we can safely assume that \mathcal{P}' itself is affine. Let $\phi: \mathcal{P} \rightarrow \mathcal{P}'$ be the morphism and set $\mathcal{A} := \phi_* \mathcal{O}_{\mathcal{P}}$. Then $\mathbf{Spec} \mathcal{A}$ is a pushout datum since ϕ is flat. Hence we obtain a factorization $\mathcal{P} \rightarrow \mathbf{Spec} \mathcal{A} \rightarrow \mathcal{P}'$ where the second morphism is affine, and the first morphism is a quasi-compact open immersion since ϕ is quasi-affine. Set $\overline{\mathcal{P}} = (\overline{T}; \overline{Y}, \overline{Z}) := \mathbf{Spec} \mathcal{A}$. We will show that in this case \mathcal{P} possesses an affine presentation which is even a Zariski presentation. Choose an open covering of Y by basic open affine subsets U_1, \dots, U_n of \overline{Y} . Then $Y_0 := \sqcup_{i=1}^n U_i$ is an affine presentation of Y , each $W_i := U_i \times_Y T$ is a basic open affine subset of \overline{T} and $W := \sqcup_{i=1}^n W_i$ is an affine presentation of T . Since $\overline{T} \hookrightarrow \overline{Z}$ is a closed immersion of affine schemes, each W_i is the restriction onto \overline{T} of a basic open affine subset $V_i \subseteq \overline{Z}$ contained in Z . Choose an affine presentation $V_0 \rightarrow Z \setminus T$, and set $Z_0 := \sqcup_{i=0}^n V_i$. Then $(T_0; Y_0, Z_0)$ is an affine presentation of \mathcal{P} . \square

Corollary 3.2.3. *Assume that $\mathcal{P}_0 \rightarrow \mathcal{P}$ is a flat morphism of Ferrand's pushout data. If \mathcal{P}_0 is affine then $\mathcal{P}_1 = \mathcal{P}_0 \times_{\mathcal{P}} \mathcal{P}_0$ admits an affine presentation.*

Proof. The base change $\mathcal{P}_1 \rightarrow \mathcal{P}_0$ of $\mathcal{P}_0 \rightarrow \mathcal{P}$ is flat, and \mathcal{P}_0 is affine. Furthermore, $\mathcal{P}_1 \rightarrow \mathcal{P}_0$ is quasi-affine since \mathcal{P}_1 is open in its schematic closure $\mathcal{P}' \hookrightarrow \mathcal{P}_0 \times \mathcal{P}_0$, and $\mathcal{P}' \rightarrow \mathcal{P}_0$ is affine. Thus, Lemma 3.2.1(ii) applies and the corollary follows. \square

3.3. Main results.

3.3.1. Equivalences of flat and étale sites. In this section we study affine Ferrand's pushouts, so $\mathcal{P} = (T; Y, Z)$ and X are affine. Let \mathbf{Fl}/X denote the category of qcqs X -flat algebraic spaces, and define $\mathbf{Ét}/X$, \mathbf{Fl}/\mathcal{P} and $\mathbf{Ét}/\mathcal{P}$ similarly. We do not know if (or when) the base change functor $\phi_{\mathbf{Fl}}$ is an equivalence, but at least it is easy to describe the only possible obstacle to this. Let $\mathbf{Fl}^{\text{pres}}/\mathcal{P}$ be the full subcategory of \mathbf{Fl}/\mathcal{P} whose objects admit an affine presentation.

Lemma 3.3.1. *The functors $\phi_{\mathbf{Ét}}$ and $\phi_{\mathbf{Fl}}$ induce equivalences of sites $\mathbf{Ét}/X \xrightarrow{\sim} \mathbf{Ét}/\mathcal{P}$ and $\mathbf{Fl}/X \xrightarrow{\sim} \mathbf{Fl}^{\text{pres}}/\mathcal{P}$.*

Proof. Recall that any object of $\mathbf{Ét}/\mathcal{P}$ admits an affine presentation by Lemma 3.2.1(ii). Now, it is easy to see that the étale case follows from the flat case, so we will consider the flat site in the sequel. If X' is in \mathbf{Fl}/X then any étale affine presentation of X' induces an étale affine presentation of $\phi_{\mathbf{Fl}}(X')$. We will show that the induced functor $\phi: \mathbf{Fl}/X \xrightarrow{\sim} \mathbf{Fl}^{\text{pres}}/\mathcal{P}$ is an equivalence.

Step 1: ϕ is faithful. Let $f, g: R \rightarrow S$ be two different morphisms in \mathbf{Fl}/X , and assume to the contrary that $\phi(f) = \phi(g)$. Then f and g coincide on the level of sets since the map $|Y| \sqcup |Z| \rightarrow |X|$ is surjective. Hence, the induced maps between the reduced spaces R^{red} and S^{red} coincide. Pick an étale affine presentation $S_0 \rightarrow S$, and consider the base changes $R \times_f S_0$ and $R \times_g S_0$ étale over R . Since over R^{red}

these spaces are isomorphic, and the étale sites over R and over R^{red} are canonically equivalent by [EGA, IV₄, 18.1.2], it follows that $R \times_f S_0$ and $R \times_g S_0$ are also isomorphic. Thus, after replacing S with S_0 and R with $R \times_f S_0$ we may assume that S is affine. Furthermore, after replacing R with an affine presentation, we may assume that R is also affine, which is a contradiction since $\text{Fl}^{\text{aff}}/X \not\cong \text{Fl}^{\text{aff}}/\mathcal{P}$.

Step 2: ϕ is full. We should show that for any pair of X -flat algebraic spaces R, S any morphism $f: \phi(R) \rightarrow \phi(S)$ in Fl/\mathcal{P} comes from a morphism $R \rightarrow S$.

Case (a): R and S are separated. Fix affine presentations $S_0 \rightarrow S$ and $R_0 \rightarrow R \times_X S_0$. Note that $R_0 \rightarrow R$ is also a presentation, and the spaces $S_1 = S_0 \times_S S_0$ and $R_1 = R_0 \times_R R_0$ are affine by the separatedness assumption. Since ϕ commutes with fibred products, f naturally lifts to a morphism $f_0: \phi(R_0) \rightarrow \phi(S_0)$. Note that f_0 induces a morphism $f_1: \phi(R_1) \rightarrow \phi(S_1)$, and f can be reconstructed from f_0, f_1 by flat descent. Since $\text{Fl}^{\text{aff}}/X \xrightarrow{\sim} \text{Fl}^{\text{aff}}/\mathcal{P}$, we can lift f_i to morphisms $h_i: R_i \rightarrow S_i$. We claim that h_i satisfy the cocycle descent condition. Indeed, this condition reduces to equality of two morphisms in Fl/X , and we know that they become equal after applying ϕ because f_i satisfy the analogous descent condition. It remains to use the fact that ϕ is faithful by Step 1. Thus, by flat descent there exists a morphism $h: R \rightarrow S$ such that $\phi(h) = f$.

Case (b): R and S are arbitrary. The argument is similar. Pick affine presentations $S_0 \rightarrow S$ and $R_0 \rightarrow R \times_X S_0$, and construct R_i, S_i, f_i as above. The only difference is that R_1 and S_1 are not necessarily affine. However, the morphism h_1 exists thanks to Case (a). The rest of the proof is identical to Case (a).

Step 3: ϕ is essentially surjective. We should show that any object $\mathcal{P}' = (T'; Y', Z')$ of $\text{Fl}^{\text{pres}}/\mathcal{P}$ lies in the essential image of ϕ .

Case (a): \mathcal{P}' is separated. Pick an affine presentation $\mathcal{P}_0 = (T_0; Y_0, Z_0)$ of \mathcal{P}' , and note that its fibred square over \mathcal{P}' is another affine presentation of \mathcal{P}' , that we denote by $\mathcal{P}_1 = (T_1; Y_1, Z_1)$. As we saw in §3.1.3, there exist X -flat pushouts X_i corresponding to the data \mathcal{P}_i . Furthermore, X_i admit natural morphisms $p_{0,1}: X_1 \rightrightarrows X_0$ and $\Delta: X_0 \rightarrow X_1$ since ϕ is full; and these morphisms define an equivalence relation because their image under ϕ does so and ϕ is faithful. Set $X' := X_0/X_1$. Then X' is an X -flat algebraic space by descent, and since $Y' = Y_0/Y_1$, $Z' = Z_0/Z_1$, $T' = T_0/T_1$ and $\phi(X_i) \xrightarrow{\sim} \mathcal{P}_i$ for $i = 0, 1$, a simple diagram chase shows that $\phi(X') \xrightarrow{\sim} \mathcal{P}'$.

Case (b): \mathcal{P}' is arbitrary. We choose an affine presentation \mathcal{P}_0 as above. This time \mathcal{P}_1 does not have to be affine, but it is separated and admits an affine presentation by Corollary 3.2.3. Therefore, $\mathcal{P}_1 \xrightarrow{\sim} \phi(X_1)$ by Case (a) and similarly to the proof of Case (a) we can construct $X' = X_0/X_1$. \square

3.3.2. Affine pushouts in the category of algebraic spaces.

Lemma 3.3.2. *If $X = Y \sqcup_T^{\text{aff}} Z$ is an affine Ferrand's pushout then $X = Y \sqcup_T Z$ in the category of all algebraic spaces. Moreover, if X' is an object of Fl/X and $(T'; Y', Z')$ is its image in Fl/\mathcal{P} then $Y' \sqcup_{T'} Z' \xrightarrow{\sim} X'$.*

Proof. Set $h_U(\mathcal{P}') := \text{Mor}(\mathcal{P}', U)$, $h_U(X') := \text{Mor}(X', U)$, etc. We should prove that the natural map $\psi: h_U(X') \rightarrow h_U(\mathcal{P}')$ is an isomorphism.

Case 1: U is affine and X' is X -separated. Pick an affine presentation $X_0 \rightrightarrows X'$ so that $X_1 = X_0 \times_X X_0$ is affine. By the separatedness assumption, the base change with respect to $\mathcal{P} \rightarrow X$ is an affine presentation $\mathcal{P}_1 \rightrightarrows \mathcal{P}_0$ of \mathcal{P}' . Then $h_U(X')$ is the equalizer of $h_U(X_0) \rightrightarrows h_U(X_1)$ and similarly for Y', Z' , and T' . Furthermore,

$h_U(X_i)$ is the equalizer of $h_U(Y_i) \times h_U(Z_i) \rightrightarrows h_U(T_i)$, $i = 0, 1$, by the universality of $X = Y \sqcup_T^{\text{aff}} Z$ in the affine category. It follows that $h_U(X')$ is the equalizer of $h_U(Y') \times h_U(Z') \rightrightarrows h_U(T')$, that is, ψ is an isomorphism.

Case 2. *U is affine.* Choose an affine presentation $X_0 \rightarrow X'$ as earlier. This time X_1 is only separated. The rest of the argument is the same, with the only difference that we invoke Case 1 to deduce that $h_U(X_i)$ is the equalizer of $h_U(Y_1) \times h_U(Z_1) \rightrightarrows h_U(T_1)$.

Case 3. *U is separated.* Fix a morphism $\alpha: \mathcal{P}' \rightarrow U$. Find an affine presentation $q_{1,2}: U_1 \rightrightarrows U_0$ of U . It induces an affine presentation $r_{1,2}: \mathcal{P}_1 \rightrightarrows \mathcal{P}_0$ of \mathcal{P}' . By affine effectivity (see Lemma 3.1.5(ii)) we also obtain surjective étale morphisms $X_1 \rightrightarrows X_0 \rightarrow X'$ and it follows from Lemma 3.3.1 that $p_{1,2}: X_1 \rightrightarrows X_0$ is a presentation of X' . The morphisms $\alpha_i: (T_i; Y_i, Z_i) \rightarrow U_i$, which are the base changes of α , induce morphisms $h_i: X_i \rightarrow U_i$ by Case 1. Note that h_i are compatible with the projections $p_{1,2}$ and $q_{1,2}$ (i.e. we have cartesian squares $h_0 \circ p_i = q_i \circ h_1$ for $i = 1, 2$) because α_i are compatible with $q_{1,2}$ and $r_{1,2}$. By descent, h_i give rise to a morphism $h: X' \rightarrow U$ which induces α by composition, i.e. $\psi(h) = \alpha$. This shows that ψ is surjective, and injectivity of ψ is much simpler. Indeed, if $h, h': X' \rightarrow U$ satisfy $\psi(h) = \psi(h')$ then their base changes $h_0, h'_0: X_0 \rightarrow U_0$ induce the same morphism $(T_0; Y_0, Z_0) \rightarrow U_0$. Hence $h_0 = h'_0$ by Case 2, and étale descent implies that $h = h'$.

Case 4. *The general case.* This time we find any separated presentation of U and argue as in Case 3, but using Case 3 instead of Case 2 as an input. \square

3.3.3. *Main result about Ferrand's pushouts.* Now we are ready to complete the theory of general Ferrand's pushouts.

Theorem 3.3.3. (i) *If $\mathcal{P} = (T; Y, Z)$ is a Ferrand's pushout datum then the following conditions are equivalent: (a) \mathcal{P} is effective, (b) \mathcal{P} admits an affine étale presentation, (c) \mathcal{P} admits an affine fppf presentation.*

(ii) *A Ferrand's pushout datum $\mathcal{P} = (T; Y, Z)$ is effective in each one of the following cases: (a) $|T|$ is discrete, (b) \mathcal{P} is flat quasi-affine over an effective Ferrand's pushout datum.*

(iii) *If $X = Y \sqcup_T Z$ is a Ferrand's pushout then the base change induces equivalences $\phi_{\text{Fl}}: \text{Fl}/X \xrightarrow{\sim} \text{Fl}^{\text{eff}}/\mathcal{P}$, where the target is the subcategory of Fl/\mathcal{P} consisting of all effective pushout data. Restriction of ϕ_{Fl} to the categories of étale or flat quasi-affine objects induces equivalences $\phi_{\text{Ét}}: \text{Ét}/X \xrightarrow{\sim} \text{Ét}/\mathcal{P}$ and $\phi_{\text{Fl}}: \text{Fl}^{\text{qa}}/X \xrightarrow{\sim} \text{Fl}^{\text{qa}}/\mathcal{P}$.*

Proof. Only the implication (c) \implies (a) requires a proof in (i). Let $\mathcal{P}_0 = (T_0; Y_0, Z_0)$ be a flat affine presentation of \mathcal{P} . By Lemma 3.3.2 $X_0 = Y_0 \sqcup_{T_0}^{\text{aff}} Z_0$ is a pushout in the category of all algebraic spaces. Moreover, if $X_1 := X_0 \times_X X_0$ and $\mathcal{P}_1 := (T_1; Y_1, Z_1) = \mathcal{P}_0 \times_{\mathcal{P}} \mathcal{P}_0$ then $\phi_{\text{Fl}}(X_1) \xrightarrow{\sim} \mathcal{P}_1$ and Lemma 3.3.2 asserts that we also have $X_1 = Y_1 \sqcup_{T_1} Z_1$. Consider the fppf quotient by flat equivalence relation (i.e. the quotient of fppf sheaves) $X = X_0/X_1$. Then X is an algebraic stack by [LMB, Th. 10.1], hence an algebraic space by [LMB, Cor. 8.1.1]. Since X is the coequalizer of $X_1 \rightrightarrows X_0$, and similarly for Y, Z, T , a simple diagram chase shows that $Y \sqcup_T Z \xrightarrow{\sim} X$.

The second assertion follows from the first and from Lemma 3.2.1, and the third assertion follows from the first and from Lemmas 3.2.1 and 3.3.1. \square

The third assertion of the theorem implies the following result that allows to compute Ferrand's pushouts flat-locally:

Corollary 3.3.4. *Assume that $\mathcal{P}_0 \rightarrow \mathcal{P}$ is a surjective flat morphism of effective Ferrand's pushout data. Then $\mathcal{P}_1 = \mathcal{P}_0 \times_{\mathcal{P}} \mathcal{P}_0$ is an effective Ferrand's pushout datum and the pushout $X = Y \sqcup_T Z$ can be described as X_0/X_1 , where $X_i = Y_i \sqcup_{T_i} Z_i$. In particular, any S -affine Ferrand's pushout is Ferrand's pushout.*

Using this we can easily extend all good properties of affine Ferrand's pushouts to the general case.

Corollary 3.3.5. *If $X = Y \sqcup_T Z$ is a Ferrand's pushout then $|Y| \sqcup_{|T|} |Z| \xrightarrow{\sim} |X|$, $T \xrightarrow{\sim} Y \times_X Z$, $Y \rightarrow X$ is a closed immersion, and the categories of flat \mathcal{O}_X -modules and flat $\mathcal{O}_{\mathcal{P}}$ -modules are naturally equivalent.*

Proof. Assume first that $\mathcal{P} = (T; Y, Z)$ is separated. Then we can find a presentation $\mathcal{P}_1 \rightrightarrows \mathcal{P}_0$ with affine \mathcal{P}_i 's, and the pushouts $X_i = Y_i \sqcup_{T_i} Z_i$ satisfy all these properties by Proposition 3.1.3. Since $X \xrightarrow{\sim} X_0/X_1$ by Corollary 3.3.4, we can use étale descent to prove the Corollary for \mathcal{P} . If \mathcal{P} is arbitrary then we choose an affine presentation $\mathcal{P}_0 \rightarrow \mathcal{P}$; then \mathcal{P}_1 is separated and we can use the same argument. \square

Remark 3.3.6. (i) It is an interesting question if one can give an explicit criterion for a Ferrand's pushout datum to be effective. We even do not know if there exist non-effective data. Note that Ferrand describes such a criterion in the category of schemes, which is, essentially, existence of affine Zariski presentation (i.e presentation in which the coverings are disjoint unions of open immersions). In particular, one obtains a source of examples which are non-effective in the category of schemes. However, many such examples turn out to be effective in the category of algebraic spaces, and it is not easy to see whether one can use this to construct a Ferrand's pushout datum which is not effective in the algebraic spaces.

(ii) It seems that the question of existence of a non-effective Ferrand's pushout datum is closely related to the affineness conjecture for Henselian schemes, see Conjecture B in [GS, Rem. 1.23(ii)].

3.3.4. Descent of properties. Various properties of morphisms descent through Ferrand's pushouts. We do not try to make a full list of these properties but only establish the case of pro-open immersions, which will be used later, and some properties that are easy to prove. A morphism of Ferrand's pushout data $f: \mathcal{P}' \rightarrow \mathcal{P}$ is a *pro-open immersion* if it is isomorphic to the filtered projective limit of a family of open immersions (at least *a priori*, this condition is stronger than the componentwise condition).

Theorem 3.3.7. (i) *Let $X = Y \sqcup_T Z$ be Ferrand's pushout, and \mathbf{P} be one of the following properties: schematically dominant, finite, quasi-finite, finite type, flat, étale, isomorphism, open immersion, pro-open immersion. Then $g': Z \rightarrow X$ satisfies \mathbf{P} if and only if $g: T \rightarrow Y$ satisfies \mathbf{P} .*

(ii) *Assume that $\psi: \mathcal{P}' \rightarrow \mathcal{P}$ is a flat morphism between effective Ferrand's pushout data, and $h: X' \rightarrow X$ is the corresponding flat morphism between the pushouts. Let \mathbf{P} denote one of the following properties: étale, open immersion, pro-open immersion. If ψ satisfies \mathbf{P} then h satisfies \mathbf{P} .*

Proof. We prove (ii) first. The claim about étaleness follows from Theorem 3.3.3(iii). This easily implies the case of open immersions because the map $|Y| \sqcup |Z| \rightarrow |X|$ is onto and an étale morphism is an open immersion if and only if its non-empty fibers are isomorphisms. Assume now that g is a pro-open immersion: $\mathcal{P}' \simeq \lim_{\alpha} \mathcal{P}_{\alpha}$ over

\mathcal{P} . Then $X' = Y' \sqcup_{T'} Z'$ is the limit of $X_\alpha := Y_\alpha \sqcup_{T_\alpha} Z_\alpha$ in the category Fl/X by Theorem 3.3.3(iii). On the other hand, $X_\alpha \rightarrow X$ are open immersions since $\mathcal{P}_\alpha \rightarrow \mathcal{P}$ are so. Thus, $X' \rightarrow X$ is a pro-open immersion.

Only inverse implications in (i) needs a proof because g is a base change of g' . Using Corollary 3.3.4, the assertion of (i) reduces to the case of affine Ferrand's pushouts. Then the first property follows from [Fer, Prop. 5.6(2)] and the next three from [Fer, Prop. 5.6(3)]. For the remaining properties we will reduce the claim to (ii) as follows: Note that in all these cases g is flat, hence so is the morphism $\psi: (T; T, Z) \rightarrow \mathcal{P}$. In addition, we claim that ψ satisfies the same property **P**. Only the case of pro-open immersions requires some explanation, but in this case $T = \lim_\alpha Y_\alpha$ for open subschemes $Y_\alpha \hookrightarrow Y$ and so $(T; T, Z) = \lim_\alpha (T; Y_\alpha, Z)$. It remains to observe that $T \sqcup_T Z \xrightarrow{\sim} Z$ and use (ii). \square

Corollary 3.3.8. *Consider a Ferrand's pushout $X = Y \sqcup_T Z$ and assume that $T \hookrightarrow Y$ is a pro-open immersion and Y is schematic at a point $y \in Y$ that possesses a unique generalization t in T . Assume that t is discrete in T and Z is schematic at t . Then X is schematic at $y \in Y \hookrightarrow X$ and the localization X_y is isomorphic to the Ferrand's pushout $Y_y \sqcup_{T_t} Z_t$.*

Proof. Note that $(T_t; Y_y, Z_t) \rightarrow \mathcal{P}$ is a pro-open immersion because $(T_t; Y_y, Z_t)$ is the projective limit of data $(T_t; Y_\alpha, Z_\alpha)$, where $Y_\alpha \subset Y$ and $Z_\alpha \subset Z$ are neighborhoods of t that are disjoint from $T \setminus t$. By Theorem 3.3.7(ii) we obtain that $Y_y \sqcup_{T_t} Z_t \rightarrow X$ is a pro-open immersion. It remains to observe that on the level of sets

$$|Y_y \sqcup_{T_t} Z_t| = Y_{\succeq y} \cup Z_{\succeq t} = X_{\succeq y}.$$

\square

4. PRÜFER ALGEBRAIC SPACES AND PAIRS

4.1. Definitions and formulations of main results.

4.1.1. *Prüfer algebraic spaces.* An integral qcqs algebraic space X is called *Prüfer* if any modification $X' \rightarrow X$ is *trivial* (i.e. is an isomorphism). An algebraic space is called *Prüfer* if it is a finite disjoint union of reduced irreducible components and each component is Prüfer. In particular, Prüfer spaces are normal since they admit no non-trivial finite modifications.

Remark 4.1.1. (i) The terminology is motivated by the fact that affine Prüfer schemes are nothing but the spectra of Prüfer domains, see §4.1.8 below.

(ii) Although it is easy to describe Prüfer schemes using the theory of Prüfer domains, it is more difficult to descend this description to algebraic spaces. The main difficulty is that not any closed subscheme of the non-Prüfer locus of a scheme X (which is a subset of X closed under specialization) may serve as the modification locus of a modification $X' \rightarrow X$. In particular, one can easily see that any spectrum of a semivaluation ring (see §4.1.3 below) does not admit modifications that modify only the closed point s , but such a spectrum is usually not Prüfer. In order to gain better control on the modification loci we are going to introduce the notion of a Prüfer pair.

4.1.2. *Prüfer pairs.* Assume that X is a qcqs algebraic space with a quasi-compact schematically dense generizing subset $U \subset |X|$. We say that (X, U) is a *Prüfer pair* if any U -modification of X is trivial.

Remark 4.1.2. (i) An integral qcqs X is Prüfer if and only if the pair (X, η) is Prüfer, where η is the generic point of $|X|$.

(ii) For a local scheme S as in Remark 4.1.1(ii) the pair $(S, S \setminus s)$ is Prüfer (whenever $S \setminus s$ is quasi-compact).

4.1.3. *Semivaluation rings.* We will see later that an important particular case of Prüfer pairs (actually, the local case) is obtained from a pair of rings as follows: By *semivaluation ring* we mean a ring A with a fixed overring B called the *semifraction ring* of A such that B is local with maximal ideal m and $A \subset B$ is the preimage of a valuation ring R of $k = B/m$. Note that $m \subset A$, $B = A_m$, and A is a valuation ring if and only if B is a valuation ring – the latter follows easily from the fact that A is a valuation ring if and only if it is integral and for any $f \in \text{Frac}(A) \setminus A$ one has that $f^{-1} \in A$.

Remark 4.1.3. (i) We use the word “semivaluation” because the valuation of R induces a semivaluation on B with kernel m , which is unique up to equivalence, and A is the ring of integers of the semivaluation.

(ii) In abstract commutative algebra, a semivaluation (ring) is often called Manis valuation (ring). We prefer the terminology of [Tem3], where semivaluation rings were used to study relative Riemann-Zariski spaces.

(iii) Note that $A \xrightarrow{\sim} B \times_k R$ and so $\text{Spec}(A)$ is the affine Ferrand’s pushout of the closed valuation subscheme $\text{Spec}(R)$ and the localization $\text{Spec}(B)$ along the point $\text{Spec}(B/m)$.

4.1.4. *Characterization of Prüfer pairs.*

Theorem 4.1.4. *Let X be a qcqs algebraic space with a quasi-compact schematically dense generizing subset $U \subset |X|$. The following conditions are equivalent:*

- (1) *There exists a scheme X' with a surjective étale morphism $f: X' \rightarrow X$ such that any point $x' \in X'$ admits a unique minimal generalization $u' \in U' := f^{-1}(U)$, and the pair $(\mathcal{O}_{U', u'}, \mathcal{O}_{X', x'})$ is a semivaluation ring.*
- (2) *For any scheme X' with étale morphism $f: X' \rightarrow X$, any point $x' \in X'$ admits a unique minimal generalization $u' \in U' := f^{-1}(U)$, and the pair $(\mathcal{O}_{U', u'}, \mathcal{O}_{X', x'})$ is a semivaluation ring.*
- (3) *A U -admissible morphism $f: Y \rightarrow X$ is flat if and only if it is U -flat, i.e. flat at any $y \in f^{-1}(U)$.*
- (4) *If $f: Y \rightarrow X$ is a separated quasi-compact U -admissible morphism such that $\overline{U} \times_X Y \rightarrow \overline{U}$ is a pro-open immersion for a pro-open subspace \overline{U} with $U \subseteq |\overline{U}|$ then f is a pro-open immersion.*
- (5) *Any U -quasi-modification $Y \rightarrow X$ is an open immersion.*
- (6) *X admits no non-trivial U -modifications, i.e. (X, U) is a Prüfer pair.*
- (7) *X admits no non-trivial U -admissible blow-ups.*
- (8) *Any finitely generated U -trivial ideal $\mathcal{I} \subset \mathcal{O}_X$ is invertible.*

In addition, if these conditions hold then U is a pro-open subspace and the pro-open immersion $i: U \hookrightarrow X$ is affine.

4.1.5. *First implications.* In this section we establish easy implications between the conditions (1)–(8). The implications (2) \Rightarrow (1) and (5) \Rightarrow (6) \Rightarrow (7) \Leftrightarrow (8) are obvious. Let us show that (3) \Rightarrow (4). Assume that f is as in (4). Then f is flat by (3) and its restriction onto the schematically dense pro-open subspace $\overline{U} \times_X Y$ is a pro-open immersion. Therefore, f itself is a pro-open immersion by the following lemma.

Lemma 4.1.5. *Assume that $f: Y \rightarrow X$ is a flat separated quasi-compact morphism of algebraic spaces, and $U \hookrightarrow Y$ is a schematically dominant pro-open immersion such that the composition $U \rightarrow X$ is a pro-open immersion. Then f is a pro-open immersion.*

Proof. By flatness of f we obtain that the pro-open immersion $U \xrightarrow{\sim} U \times_X U \rightarrow Y \times_X Y$ is schematically dominant. Therefore, the diagonal closed immersion $Y \rightarrow Y \times_X Y$ is schematically dominant, hence an isomorphism. Thus, f is a monomorphism, hence a pro-open immersion by Theorem 2.4.2. \square

Next, we show that (4) \Rightarrow (5). Let $f: Y \rightarrow X$ be a U -quasi-modification. Then it is a pro-open immersion by (4). By the definition of U -quasi-modification, there exists an open neighborhood \overline{U} of U such that $f \times_X \overline{U}$ is an open immersion. Set $\overline{Y} := \overline{U} \sqcup_{\overline{U} \times_X Y} Y$ – the usual Zariski gluing. Clearly, $\overline{f}: \overline{Y} \rightarrow X$ is separated, hence a pro-open immersion. Thus, \overline{f} is a schematically dominant pro-open immersion of finite type, hence an open immersion by Proposition 2.4.6(ii), which implies that f is an open immersion.

Finally, we note that if (1) is satisfied then $X'_{\leq x'} \cap U' = X'_{\leq u'}$, and therefore Proposition 2.4.11 implies that $U \rightarrow X$ is an affine pro-open immersion. Our proof of the remaining claims will proceed by establishing the following implications: (8) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3). We will be occupied with the proof of these implications until the end of §4, but let us first deduce some corollaries and connect this theory to some well known results about rings.

4.1.6. *Prüfer algebraic spaces.* Taking U to be the generic point of an integral algebraic space one immediately obtains a similar description of Prüfer spaces:

Corollary 4.1.6. *Let X be an integral qcqs algebraic space. The following conditions are equivalent:*

- (1) *There exists a scheme X' with a surjective étale morphism $X' \rightarrow X$ such that for any point $x' \in X'$ the local ring $\mathcal{O}_{X', x'}$ is a valuation ring.*
- (2) *For any scheme X' with étale morphism $X' \rightarrow X$ and a point $x' \in X'$ the local ring $\mathcal{O}_{X', x'}$ is a valuation ring.*
- (3) *A morphism $Y \rightarrow X$ is flat if and only if it is η -admissible, where η is the generic point of X .*
- (4) *If $f: Y \rightarrow X$ is a separated quasi-compact morphism such that Y is integral, and f induces an isomorphism of generic points then f is a pro-open immersion.*
- (5) *Any quasi-modification $Y \rightarrow X$ is an open immersion.*
- (6) *X admits no non-trivial modifications, i.e. X is a Prüfer space.*
- (7) *Any non-empty blow up of X is an isomorphism.*
- (8) *Any non-zero finitely generated ideal $\mathcal{I} \subset \mathcal{O}_X$ is invertible.*

4.1.7. *Étale-locality of Prüferness.* As an immediate corollary of conditions (1) and (2) from Theorem 4.1.4 we obtain the following important result:

Corollary 4.1.7. *Let $f: X' \rightarrow X$ be an étale morphism between qcqs algebraic spaces, $U \subset |X|$ be a quasi-compact schematically dense generizing subset, and $U' := f^{-1}(U)$. If (X, U) is a Prüfer pair then (X', U') is a Prüfer pair, and the converse is true whenever f is surjective. In particular, for X and X' we have that if X is Prüfer then X' is Prüfer and the converse is true for surjective f .*

4.1.8. *Connection to the theory of Prüfer rings.* If $X = \operatorname{Spec}(A)$ is an integral affine scheme then X is Prüfer if and only if A is a Prüfer domain. The rings of the latter type were intensively studied in abstract commutative algebra. In particular, there are many other conditions equivalent to Prüferness of A (or X). For example, 14 such conditions are given in [Bou, Ch. VII, §2, Exercise 12], including our (1) and (8) from Corollary 4.1.6. Similarly, if $X = \operatorname{Spec}(A)$ is affine then a pair (X, U) is Prüfer if and only if $U = \operatorname{Spec}(B)$ is affine, $A \hookrightarrow B$ and the pair (A, B) is a Prüfer pair (or normal pair), as defined, for example, in [Dav].

Thus, in the affine case our theory describes a classical algebraic object from a geometric point of view, but our geometric definition of Prüferness does not appear among the 14 equivalent algebraic definitions of [Bou]. The advantage of the geometric approach is that it makes sense for global objects and for stacks. In principle, one could use the algebraic theory as a slight shortcut for proving Theorem 4.1.4, but in order to give geometric arguments, that we plan to generalize to stacks in further works, we prefer to avoid the use of the theory of Prüfer rings as much as we can.

4.2. **Proofs.** In the first two sections of §4.2 we will prove Theorem 4.1.4 in the case when X is a local scheme (this is the case, where one could shorten some arguments by quoting results on Prüfer pairs of rings). The case when X is a scheme is deduced in §4.2.3 using the fact that one can extend ideals from open subschemes. Finally, the general case is done in §4.2.4, and it requires a more subtle argument because one cannot descent arbitrary ideals through étale presentations $X' \rightarrow X$.

4.2.1. *Étale-locality.* In this section we will prove the equivalence (1) \Leftrightarrow (2) in Theorem 4.1.4. Actually, this reduces to the following Lemma:

Lemma 4.2.1. *Assume that $f: X' \rightarrow X$ is a local-étale morphism of local schemes, where $X = \operatorname{Spec}(A)$ and $X' = \operatorname{Spec}(A')$. Assume, in addition, that $U \subset X$ is a generizing subset and $U' = f^{-1}(U)$.*

(i) *If $U = \operatorname{Spec}(B)$ where $A \hookrightarrow B$ and (B, A) is a semivaluation ring then $U' = \operatorname{Spec}(B')$ where $A' \hookrightarrow B'$ and (B', A') is a semivaluation ring.*

(ii) *The converse of (i) is true whenever f is surjective (i.e. the homomorphism $A \rightarrow A'$ is local).*

Proof. We will treat both cases in the same fashion. First, we claim that in both cases we may assume that $U = \operatorname{Spec}(B)$ and $U' = \operatorname{Spec}(B')$ are local schemes with closed points x and x' , respectively. In case (i) we have $U = X_{\geq x}$. If $f^{-1}(x) = \emptyset$ then $U' = X'$ and the lemma follows, otherwise $f^{-1}(x) = \{x'\}$ by Lemma 2.2.1 applied to the integral closed subscheme of X whose generic point is x . Therefore, $U' = f^{-1}(U) = X'_{\geq x'}$. Below we will assume that $f^{-1}(x) = \{x'\}$. In case (ii), $U = f(U') = f(X'_{\geq x'}) = X_{\geq f(x')}$ since f is a generizing map (see §2.2.2).

Let $m \subset A$ and $m_1 = mB$ be the ideals corresponding to x , and let $m' \subset A'$ and $m'_1 = m'B'$ be the ideals corresponding to x' . Then $m \otimes_A A' \xrightarrow{\sim} mA' = m'$ and $m_1 \otimes_B B' \xrightarrow{\sim} m_1 B' = m'_1$ since f is essentially étale. By our assumptions, $m = m_1$ in case (i), and $m' = m'_1$ and f is faithfully flat in case (ii). So, in both cases $m = m_1$ and $m' = m'_1$ because the chain of inclusions $m \subseteq m_1 \subseteq A \subseteq B$ is taken to $m' \subseteq m'_1 \subseteq A' \subseteq B'$ by tensoring with A' over A . In particular, the local ring $R' = A'/m'$ is essentially étale over the local ring $R = A/m$.

Since A is the preimage of R in B and A' is the preimage of R' in B' , it remains to prove that if R is a valuation ring then so is R' , and the converse is true when the homomorphism $R \rightarrow R'$ is local. In case (i), this follows from Lemma 2.2.1. In case (ii), R is integrally closed and $K = \text{Frac}(R)$ is a field since R', K' are so and the map is essentially étale and surjective. Thus, $R = R' \cap K$ since $R \rightarrow R'$ is a local homomorphism. Hence R is a valuation ring. \square

By §4.1.5, it now suffices to prove the implications (8) \Rightarrow (1) and (2) \Rightarrow (3) in Theorem 4.1.4.

4.2.2. *Local case.* We will need the following simple Lemma.

Lemma 4.2.2. *Let A be a local ring, and $\{I_\alpha\}$ be a family of ideals in A such that $\sum I_\alpha$ is principal. Then $\{I_\alpha\}$ has a greatest element I_{α_0} , in particular $I_{\alpha_0} = \sum I_\alpha$.*

Proof. Pick a generator $f \in \sum I_\alpha$, $f = \sum_{i=1}^k f_{\alpha_i}$ for some $f_{\alpha_i} \in I_{\alpha_i}$. Then, $\frac{f_{\alpha_i}}{f} \in A$ for all i and $1 = \sum_{i=1}^k \frac{f_{\alpha_i}}{f}$. Hence, at least one of them is invertible, since A is local. Thus, there exists i such that $f \in I_{\alpha_i}$, which implies the lemma. \square

Proposition 4.2.3. *Theorem 4.1.4 holds whenever $X = \text{Spec}(A)$ is an integral local scheme.*

Proof. By §4.2.1, we shall prove the implications: (2) \Rightarrow (3) and (8) \Rightarrow (1).

(2) \Rightarrow (3): We shall prove that a U -flat U -admissible morphism $f: Y \rightarrow X$ is flat. The question is flat-local on Y so we can assume that Y is affine, say $Y = \text{Spec}(C)$. The assumption of (2) applied to the identity map $X \rightarrow X$ implies that $U = \text{Spec}(B)$ is a local scheme with closed point u . Furthermore, (B, A) is a semivaluation ring, so the maximal ideal $m \subset B$ is contained in A , $B = A_m$, and $R = A/m$ is a valuation ring of $k(u) = B/m$. Then $A = B \times_{k(u)} R$, and by [Ryd3, Prop. 6.7(vii)], C is A -flat if and only if $C \otimes_A R$ is R -flat. Since R is a valuation ring, an R -module is flat if and only if it has no R -torsion. It remains to note that $C \otimes_A R = \mathcal{O}(Y \times_X T)$ has no R -torsion because $Y \times_X T \rightarrow T$ is u -admissible, since $u = U \times_X T$.

(8) \Rightarrow (1): First we claim that U is local. Assume to the contrary that $u, u' \in U$ are two distinct closed points, and consider their closures $Z, Z' \subset X$. Clearly, $Z \cap Z' \cap U = \emptyset$, hence also $Z \cap Z' \cap V = \emptyset$ for some qcqs open neighborhood of U . By Lemma 4.2.4 below, there exist finite type ideals $\mathcal{I}, \mathcal{I}' \subset \mathcal{O}_V$ with disjoint supports such that $Z \cap V \subseteq V(\mathcal{I})$ and $Z' \cap V \subseteq V(\mathcal{I}')$. By [Ryd1, Th. A], these ideals can be extended to finite type ideals $\mathcal{J}, \mathcal{J}' \subset \mathcal{O}_X$, and then the ideal $\mathcal{K} = \mathcal{J} + \mathcal{J}'$ is U -trivial, hence principal by the assumption of (8). By Lemma 4.2.2, either $\mathcal{J} = \mathcal{K}$ or $\mathcal{J}' = \mathcal{K}$, which is a contradiction, since Z and Z' are disjoint. Thus, the quasi-compact generizing subset U contains at most one closed point. Hence $U = U_{\geq u} = \text{Spec}(B)$, where $B = \mathcal{O}_{X,u}$. In particular, $B = A_p$, where $p = m \cap A$ and m is the maximal ideal of B .

It remains to prove that (B, A) is a semivaluation ring. First, we claim that $p = m$. Indeed, any $g \in m$ is of the form $s^{-1}f$ with $s \in A \setminus p$ and $f \in p$. The ideal $sA + p$ is U -trivial, hence principal. Therefore, $p \subseteq As$ by Lemma 4.2.2, since $s \notin p$. This shows that $g = s^{-1}f \in m \cap A = p$ as claimed. In particular, $A = \pi^{-1}(R)$, where $\pi: B \rightarrow B/m$ denotes the natural projection and $R := \pi(A) \subset B/m$. Now, it remains to check that R is a valuation ring. For any choice of $f, g \in A \setminus p$ the ideal $J := Af + Ag$ is U -trivial, hence principal. Thus, either $f \in gA$ or $g \in fA$ by Lemma 4.2.2, and it follows that for any pair of non-zero elements $a, b \in R$ either $a \in bR$ or $b \in aR$. So, R is a valuation ring and we are done. \square

Lemma 4.2.4. *Let X be a qcqs scheme, and $Z, Z' \subset X$ be closed subschemes such that $Z \cap Z' = \emptyset$. Then there exist finite type ideal sheaves $\mathcal{I}, \mathcal{I}' \subseteq \mathcal{O}_X$ such that $Z \subseteq V(\mathcal{I})$, $Z' \subseteq V(\mathcal{I}')$, and $V(\mathcal{I}) \cap V(\mathcal{I}') = \emptyset$.*

Proof. Let $\mathcal{J}, \mathcal{J}' \subseteq \mathcal{O}_X$ be the ideal sheaves of Z and Z' . By [EGA I, 6.9.9], both \mathcal{J} and \mathcal{J}' are filtered unions of finitely generated subideals \mathcal{J}_α and \mathcal{J}'_β , respectively. Thus, $\cap_\alpha V(\mathcal{J}_\alpha)$ and $\cap_\beta V(\mathcal{J}'_\beta)$ are disjoint, and, by the quasi-compactness of X , there exist α_0, β_0 such that $V(\mathcal{J}_{\alpha_0}) \cap V(\mathcal{J}'_{\beta_0}) = \emptyset$. \square

4.2.3. *Scheme case.*

Proposition 4.2.5. *Theorem 4.1.4 holds whenever X is a scheme.*

Proof. By §4.2.1 we shall establish two implications:

(2) \Rightarrow (3): We will give an argument that works for an arbitrary algebraic space X . Assume that (2) holds, and let $f: Y \rightarrow X$ be a U -admissible U -flat morphism. Choose an étale presentation $h: X' \rightarrow X$, and for any point $x \in X'$ consider the localization $X'_x = \text{Spec}(\mathcal{O}_{X',x})$. It is sufficient to prove that the base change $f_x = f \times_X X'_x$ is flat (cf. the proof of Proposition 2.4.4). Let U'_x be the preimage of U in X'_x . Then the pair (X'_x, U'_x) satisfies (1), and f_x is a U'_x -admissible U'_x -flat morphism. But we already proved that (1) \Rightarrow (2) \Rightarrow (3) for local schemes, and so f_x is flat.

(8) \Rightarrow (1): Let us show that the pair $(X', f) = (X, id)$ satisfies the requirements of (1). Assume to the contrary that there exists $x \in X$ such that (X_x, U_x) is not obtained from a semivaluation ring, where $X_x := \text{Spec}(\mathcal{O}_{X,x})$ and $U_x := U \cap X_x$. By Proposition 4.2.3, there exists a U_x -trivial finitely generated ideal $I_x \subset \mathcal{O}_{X,x}$ which is not invertible. It follows easily from [EGA I, 6.9.7] that I_x can be extended to a finitely generated U -trivial ideal $\mathcal{I} \subset \mathcal{O}_X$. But then \mathcal{I} is not invertible, which is a contradiction. \square

4.2.4. *General algebraic spaces.* In this section we will prove Theorem 4.1.4 for a general algebraic space X . Since the implication (2) \Rightarrow (3) has been proven for such X in the previous subsection, it remains to establish the implication (8) \Rightarrow (1).

Assume to the contrary that (8) holds but (1) is false for some X . Consider an affine presentation $f: X' \rightarrow X$ with a point $x' \in X'$ that violates (1). Set $U' := f^{-1}(U)$. To get a contradiction, we are going to construct an ideal $\mathcal{I} \subset \mathcal{O}_X$ that violates (8). The main task is to construct such an ideal on an open subspace, because we can then extend it by [Ryd1, Th. A] to X . Thus, in the sequel, we may shrink X whenever is necessary (note that if $X_0 \subset X$ is quasi-compact then $U_0 = U \cap X_0$ is also quasi-compact by quasi-separatedness of X).

Since the identity morphism violates (1) for the scheme X' , and the theorem has been proven in this case, there exists a non-invertible finitely generated U' -trivial

ideal $\mathcal{I}' \subset \mathcal{O}_{X'}$. Although \mathcal{I}' need not be descendable to X , it will help us to produce \mathcal{I} as required. Let $Z' \subset X'$ be the closed locus where \mathcal{I}' is not invertible. Then $Z' \neq \emptyset = Z' \cap U'$. Since f is quasi-finite, the image $Z := f(Z') \subset X$ is locally closed. We claim that after shrinking X we may assume that $Z \neq \emptyset$ is closed. Indeed, the open subspace $V = X \setminus (\overline{Z} \setminus Z)$ is a filtered union of quasi-compact open subspaces V_α . Plainly, $Z \cap V_\alpha$ is closed in V_α and $Z \cap V_\alpha$ is non-empty for α large enough. Thus, we replace X with an appropriate V_α .

Pick a generic point $\eta \in Z$. Then there exists an étale morphism $g: X'' \rightarrow X$ strictly étale over η . Plainly, g is strictly étale over a neighborhood of η in Z . Thus, after shrinking X again as above, we may assume that g is surjective and strictly étale over Z . Finally, since we may increase U as long as it stays quasi-compact and disjoint from Z , we may assume that U is generizing and g is strictly étale over $X \setminus U$ thanks to the following:

Claim 4.2.6. *There exists a generizing quasi-compact subset $U \subseteq V$ disjoint from Z and such that g is strictly étale over $X \setminus V$.*

We postpone the proof of the claim, and first finish the proof of the Theorem. Set $U'' := g^{-1}(U)$ and $\tilde{X} := X' \times_X X''$, and let $\tilde{U} \subset \tilde{X}$ be the preimage of U . Since Theorem 4.1.4 and Corollary 4.1.7 are already established for schemes, and the pair (X', U') is not Prüfer, it follows that the pair (\tilde{X}, \tilde{U}) is not Prüfer. Therefore, (X'', U'') is not Prüfer either, and there exists a U'' -trivial, finitely generated, non-invertible ideal \mathcal{I}'' on X'' . We claim that \mathcal{I}'' is descendable to X , which simply follows from the fact that $X'' \times_X X''$ is a disjoint union of the diagonal (isomorphic to X'') and an open subscheme whose images in X'' belong to U'' . So, we obtain an ideal $\mathcal{I} \subset \mathcal{O}_X$ with $\mathcal{I}\mathcal{O}_{X''} = \mathcal{I}''$, and étale descent implies that \mathcal{I} violates (8). \square

Proof of the Claim. Let X_1 be the locus of all points $x \in X$ over which the rank of g is at least two. Then $x \in X_1$ if and only if g is not strictly étale over x . Note that X_1 is open and quasi-compact because it is the image of the open quasi-compact set $W \subset X'' \times_X X''$, which is the union of the components different from the diagonal. Thus, we can chose $V = U \cup X_1$. \square

4.3. Some other properties of Prüfer pairs. In this section, we describe few more properties of Prüfer pairs and spaces. They all are simple corollaries of Theorem 4.1.4.

4.3.1. Topology.

Proposition 4.3.1. *If X is an integral Prüfer algebraic space then $|X|$ is a Zariski tree. In particular, if X is a valuation algebraic space, i.e. X is a Prüfer space with a unique closed point, then $|X|$ is a Zariski chain.*

Proof. Choose an étale presentation $f: X' \rightarrow X$. Then $X'_{\geq x'}$ is a Zariski chain for any $x' \in X'$ since it is the spectrum of a valuation ring; and since f is generizing, $X'_{\geq f(x')} = f(X'_{\geq x'})$ is also a Zariski chain. We conclude that $|X|$ is a Zariski tree by surjectivity of f . \square

4.3.2. Compatibility with immersions.

Proposition 4.3.2. *Assume that X is a qcqs algebraic space and $i: X' \hookrightarrow X$ is a quasi-compact pro-open immersion. Assume also that $U \subset |X|$ is a generizing subset such that (X, U) is a Prüfer pair. Then (X', U') is a Prüfer pair, where*

$U' = U \cap |X'|$. In particular, quasi-compact pro-open subspaces of a Prüfer algebraic space are Prüfer.

Proof. By étale-locality of Prüferness we can replace X with its étale presentation and then this follows from Theorem 4.1.4 (1). \square

Proposition 4.3.3. (i) Assume that X is a qcqs algebraic space and $i: X' \hookrightarrow X$ is a closed immersion. Assume that $U \subset |X|$ is a generizing subset such that (X, U) is a Prüfer pair and $U' = U \cap |X'|$ is schematically dense in X' . Then (X', U') is a Prüfer pair.

(ii) Any integral closed subscheme of a Prüfer algebraic space is Prüfer.

Proof. Both assertions are proved similarly, so we will deal only with (i). As in the previous proof, we can replace X with its affine presentation. So, $X = \text{Spec}(A)$ and $X' = \text{Spec}(A')$. Assume to the contrary that the pair (X', U') is not Prüfer. Then there exists a U' -trivial finitely generated non-invertible ideal $I' \subset A'$. Lift this ideal to A , that is, find a finitely generated ideal $I \subset A$ with $IA' = I'$. Obviously, I is non-invertible, so to get a contradiction it is enough to show that I could be chosen to be U -trivial. Note that $A' = A/J$ and $I + J$ is U -trivial. It follows from the quasi-compactness of U that already some intermediate finitely generated ideal $I \subseteq I_1 \subseteq I + J$ is U -trivial. So, I_1 is a required lifting of I and we are done. \square

4.3.3. Generizing subsets.

Proposition 4.3.4. Let (X, U) be a Prüfer pair and V be a quasi-compact generizing subset containing U . Then V is a pro-open subspace and the pro-open immersion $V \hookrightarrow X$ is affine. In particular, any quasi-compact generizing subset of a Prüfer space is a pro-open subspace with affine immersion morphism.

Proof. The pair (X, V) is Prüfer, hence this follows from the last assertion of Theorem 4.1.4. \square

4.3.4. *Composing Prüfer spaces.* It is well known that if C is a valuation ring and B is a valuation ring of the residue field $K = C/m$ then the preimage of B in C is a valuation ring A composed from C and B . Similarly, a semivaluation ring is composed from a local ring (its semifraction field) and a valuation ring. These observations can be generalized to Prüfer pairs as follows:

Proposition 4.3.5. Assume that (Z, U) is a Prüfer pair, $T = \sqcup_{i=1}^n t_i \subset Z$ is a disjoint union of closed Zariski points, and $Y = \sqcup_{i=1}^n Y_i$, where each Y_i is an integral Prüfer algebraic space with generic point t_i . Then U can be naturally identified with a generizing subset of the Ferrand's pushout $X = Y \sqcup_T Z$ and the pair (X, U) is Prüfer.

Proof. Note that in our case Ferrand's pushout exists by Lemma 3.2.1(i) and Theorem 3.3.3(i). By Theorem 3.3.7(ii) $Z \rightarrow X$ is a schematically dominant pro-open immersion, hence we can view U as a schematically dense generizing subset of $|X|$. Assume, to the contrary, that (X, U) is not Prüfer and let $f: X' \rightarrow X$ be a non-trivial U -modification. Since (Z, U) is Prüfer, f is an isomorphism over Z . So, f is an isomorphism over t_i 's, and hence it is an isomorphism over Y_i 's. Thus, f is a schematically dominant finite morphism whose fibers are isomorphisms. It follows easily (e.g. by reduction to the affine case) that f is an isomorphism. The contradiction shows that (X, U) is Prüfer. \square

4.3.5. *Disassembling Prüfer spaces.* The following proposition generalizes the well known fact that a valuation ring of finite height $h > 1$ is composed from valuation rings of smaller heights.

Proposition 4.3.6. *Let (X, U) be a Prüfer pair, and $T = \{t_1, \dots, t_n\} \subset X$ be a discrete subset of points such that $T \cap U$ is closed in U . Let $Y \subset X$ be the Zariski closure of T provided with the reduced subspace structure. Set $Z := X \setminus \bigcup_{i=1}^n X_{\prec t_i}$. Then $Y \sqcup_T Z \xrightarrow{\sim} X$ is a Ferrand's pushout.*

Proof. By Propositions 4.3.2 and 4.3.3, (Z, U) and Y are Prüfer. Thus, by Proposition 4.3.5, there exists Ferrand's pushout $X' = Y \sqcup_T Z$, and the pair (X', U) is Prüfer. Furthermore, X' is X -affine since Y and Z are so. Thus, the morphism $f: X' \rightarrow X$ is separated, quasi-compact, U -admissible, and induces an isomorphism over U . Hence f is a pro-open immersion by condition (4) of Theorem 4.1.4. But $|X| = |Y| \cup |Z|$ by the construction. Thus, f is surjective, hence an isomorphism. \square

4.3.6. *Separatedness.*

Proposition 4.3.7. *Assume that $f: X \rightarrow S$ is a morphism of schemes whose source is integral and Prüfer. Then f is not separated if and only if there exist points $x, y \in X$ such that $f(x) = f(y)$ and $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ coincide as subrings of $k(X)$. In particular, X itself is not separated if and only if $\mathcal{O}_{X,x}$ coincides with $\mathcal{O}_{X,y}$ for two distinct points of X .*

Proof. Actually, this is a version of the valuative criterion of separatedness for f and X . The assertion is local on S , hence we can assume that S is affine. Then this reduces to the second assertion (about separatedness of X), so we should only establish that. It is easy to see that in the usual valuative criterion for an integral scheme Y , it suffices to test only the valuations of $k(Y)$ (this is the claim of [Har, Ex. II.4.5(c)]; also, this follows from the results of [Tem3, §3.2]). Thus, X is separated if and only if for any valuation ring \mathcal{O} of $k(X)$ there exists at most one morphism $f: \text{Spec}(\mathcal{O}) \rightarrow X$ compatible with the generic point. To give f is the same as to choose a point $x \in X$ such that \mathcal{O} dominates the local ring $\mathcal{O}_{X,x}$ in $k(X)$ (i.e. the embedding homomorphism $f^*: \mathcal{O}_{X,x} \rightarrow \mathcal{O}$ is local). Since X is Prüfer, $\mathcal{O}_{X,x}$ is a valuation ring, and hence any domination homomorphism f^* as above is an isomorphism. In particular, $\text{Spec}(\mathcal{O})$ admits two different morphisms to X if and only if $\mathcal{O} = \mathcal{O}_{X,x} = \mathcal{O}_{X,y}$ for two different points $x, y \in X$. \square

5. VALUATION ALGEBRAIC SPACES

5.1. SLP algebraic spaces.

5.1.1. *Basic definitions.* A valuation algebraic space X is a qcqs Prüfer algebraic space with unique closed point. By *height* of X we mean the topological dimension of $|X|$. In general, a valuation algebraic space does not have to possess an étale presentation U which is local (hence a valuation scheme). So, it is more convenient to work with the wider class of *semi-local Prüfer algebraic spaces (SLP spaces)*, whose objects are qcqs Prüfer algebraic spaces with finitely many closed points. The following Lemma follows easily from Corollary 4.1.7:

Lemma 5.1.1. *A qcqs algebraic space X is SLP if and only if one (hence any) presentation of X is so.*

5.1.2. Examples. We will prove in Proposition 5.1.5 that any separated SLP space T is an affine scheme, and so T is the spectrum of a semi-local Prüfer ring. To gain some intuition on non-schematic valuation spaces we consider few basic examples, which reflect the general behavior of such creatures. We will treat three different examples in a uniform way.

Let \mathcal{O} be a DVR with field of fractions K . Assume that there exist quadratic extensions K_1 , K_2 , and K_3 of K such that the valuation splits in K_1 , $f_{K_2/K} = 2$, and $e_{K_3/K} = 2$. For example, they exist for $\mathcal{O} = \mathbf{Z}_{(p)}$. For any i , let $\mathcal{O}_i \subset K_i$ be the integral closure of \mathcal{O} . Set $X := \text{Spec}(\mathcal{O}) = \{\eta, s\}$ and $U_i := \text{Spec}(\mathcal{O}_i)$. Then U_1 has two closed points, and U_2 and U_3 are local schemes. Set $R_i := U_i \times_X U_i$. Then $R_i \rightrightarrows U_i$ are flat equivalence relations with $U_i/R_i \xrightarrow{\sim} X$; and they are étale for $i = 1, 2$. Plainly, $R_i = R_i^+ \cup R_i^-$, where R_i^+, R_i^- are irreducible components and R_i^+ denotes the diagonal of the relation R_i . Furthermore, the projections $R_i^+ \rightarrow U_i$ and $R_i^- \rightarrow U_i$ are isomorphisms, $R_1^+ \cap R_1^- = R_2^+ \cap R_2^- = \emptyset$, and $R_3^+ \cap R_3^-$ is the closed point of both R_3^+ and R_3^- . Let η_i^- be the generic point of R_i^- .

Set $R'_i := R_i^+ \sqcup \eta_i^-$. Then the natural monomorphism $R'_i \rightarrow R_i$ is a locally closed immersion if $i \leq 2$ and is a bijective non-isomorphism for $i = 3$. Plainly, the induced projections $R'_i \rightrightarrows U_i$ are étale equivalence relations for any i , so $X_i := U_i/R'_i$ are algebraic spaces. Furthermore, X_i are non-separated SLP spaces, and the natural morphisms $f_i: X_i \rightarrow X$ induced from the morphisms of charts are isomorphism over η . The Zariski fibers $f_i^{-1}(s)$ consist of: two points s_1, s'_1 with residue field $k(s)$ for $i = 1$, one point s_2 with larger residue field for $i = 2$, and one point s_3 with residue field $k(s)$ for $i = 3$. Plainly, X_1 is a scheme isomorphic to X with doubled closed point, X_2 is a valuation algebraic space which is locally separated but not separated, and X_3 is a valuation algebraic space which is not locally separated. In particular, X_i are not schemes for $i = 2, 3$.

Remark 5.1.2. The first example is classical (but described with an étale chart). The second example is essentially Knutson's example of a line with "twisted" doubled origin, so that non-separatedness of X_2 is hidden in "doubling" the residue field at the origin. The non-separatedness of X_3 is in "replacing" its tangent space with the tangent space of a ramified extension. Note that Knutson considered another basic example of not locally separated algebraic space, namely, the line with "doubled" tangent space at the origin. The latter is not normal and its "speciality" can be eliminated by normalization. In particular, such example is irrelevant in the study of valuation algebraic spaces. However, we have seen that not locally separated valuation algebraic spaces exist as well.

Remark 5.1.3. It is easy to see that in the above examples $X_i = U_i \sqcup_{\eta_i} \eta$ where $\eta_i = \text{Spec}(K_i)$. On the other hand it is also easy to see that the schematic pushout exists and is equal to X_i for $i = 1$ and to X for $i = 2, 3$. In the latter case, schematic and affine pushouts coincide and are different from the pushout in the category of algebraic spaces.

5.1.3. Topology.

Lemma 5.1.4. *If X is an SLP space then $|X|$ is a Zariski tree with finitely many closed points. In particular, any generizing subset $U \subset |X|$ is of the form $(\cup_{i=1}^n X_{\succeq x_i}) \cup (\cup_{i=n+1}^m X_{\succ x_i})$, where the set $\{x_1, \dots, x_m\}$ is discrete, and U is quasi-compact if and only if there exists such representation with $m = n$.*

Proof. Note that since X is quasi-compact, it is the finite union of Zariski chains connecting the generic point of X to its closed points. Therefore, the claim easily reduces to the case of Zariski chains, cf. §2.2.4. \square

We aware the reader that the sets $X_{>x}$ do not have to be open in a general SLP space.

5.1.4. Affine SLP spaces.

Proposition 5.1.5. *Let $f: X \rightarrow S$ be a morphism of algebraic spaces.*

(i) *Assume that X is SLP, then f is separated if and only if f is affine. In particular, any separated SLP space is an affine scheme.*

(ii) *Assume that X is a valuation space, f is separated, and S is a scheme then X is affine. In particular, the following conditions on a valuation space are equivalent: X is separated, X is a scheme, X is the spectrum of a valuation ring.*

Proof. The assertion of (i) is étale local on X , and the assertion of (ii) is Zariski local on the image of the closed point of X under f . Therefore, in both cases it is sufficient to prove that if X is a separated SLP then it is an affine scheme. Without loss of generality we may assume that X is irreducible. Let η be its generic point. Pick an affine étale presentation $U \rightarrow X$, and set $R := U \times_X U$. Let U_1, \dots, U_r be the irreducible components of U , and $\eta_U = \sqcup_{i=1}^r \eta_i$ be the scheme of its generic points. Then $\mathcal{O}(U) = \prod_{i=1}^r \mathcal{O}(U_i)$, $\mathcal{O}(\eta_U) = \prod_{i=1}^r k(\eta_i)$, and each $\mathcal{O}(U_i)$ is an intersection of finitely many valuation rings with field of fractions $k(\eta_i)$. In particular, $A := \mathcal{O}(U) \cap k(\eta) \subseteq \mathcal{O}(\eta_U)$ is a finite intersection of valuation rings of $k(\eta)$. We will prove the Proposition by showing that X is isomorphic to $X' := \text{Spec}(A)$.

By [Bou, Ch. VI, §7.1, Prop. 1 and 2], A is a semi-local Prüfer ring and the morphism $g: U \rightarrow X'$ is surjective. Furthermore, g is flat by [Bou, Ch. VI, §3.6, Lemma 1], and it follows that $\eta_U \times_{\eta} \eta_U$ is schematically dense in $R' := U \times_{X'} U$. So, both R and R' are the schematic closures of $\eta_U \times_{\eta} \eta_U$ in $U \times U$ since f is separated, which implies $R = R'$. Thus, the flat base change of g with respect to itself is étale; hence g is étale by flat descent, and so $X \xrightarrow{\sim} U/R \xrightarrow{\sim} X'$. \square

5.1.5. *Schematic morphisms and separatedness.* Note that Proposition 5.1.5 immediately implies that any valuation algebraic space T that admits a separated morphism to a scheme is itself separated, hence a valuation scheme. We will need the following relative version of this fact.

Lemma 5.1.6. *Assume that $Y \rightarrow Z$ is a schematic morphism between algebraic spaces and $T \rightarrow Y$ is a separated morphism whose source is a valuation algebraic space. Then the composition $T \rightarrow Z$ is separated.*

Proof. Assume to the contrary that $T \rightarrow Z$ is not separated. Note that by Propositions 4.3.1 the generizing subsets of T are totally ordered by inclusion, and each such quasi-compact subset is a pro-open subspace by Proposition 4.3.4. Therefore, there exists a minimal generizing subset T_0 such that the morphism $T_0 \rightarrow Z$ is non-separated. By its definition, T_0 is not a union of its proper open subspaces. Thus, T_0 is quasi-compact, hence a pro-open subspace of the form $T_{\geq t}$ (see §2.2.4). Since T_0 is itself a valuation subspace by Proposition 4.3.2, we can replace T with T_0 and assume in the sequel that t is the closed point of T and the morphism $T \setminus \{t\} \rightarrow Z$ is separated.

Let $y \in Y$ and $z \in Z$ be the images of t . By [Knu, Th. II.6.4], there exists a scheme Z' with an étale morphism $Z' \rightarrow Z$ which is strictly étale over z . Consider the base change $T' \rightarrow Y' \rightarrow Z'$ of $T \rightarrow Y \rightarrow Z$. Then $T' \rightarrow Y'$ is a separated morphism from an SLP space to a scheme, hence T' is an SLP scheme by Proposition 5.1.5. Clearly, the preimage of t in T' consists of a single point t' , the morphism $T' \setminus \{t'\} \rightarrow Z'$ is separated, and the morphism $T' \rightarrow Z'$ is not separated. It follows from Proposition 4.3.7 that $\mathcal{O}_{T', t'} = \mathcal{O}_{T', t''}$ for some other point $t'' \in T'$. But it is easy to see that the latter is impossible because the flat morphism $h: T' \rightarrow T$ takes t'' to a strict generalization of $t = h(t')$. \square

5.1.6. Valuative obstacle for existence of schematization. It is well known that any separated integral qcqs algebraic space X can be *schematized* by a non-empty blow up $X' \rightarrow X$. If X is of finite type over \mathbf{Z} then this follows from the Chow lemma, and the general case reduces to this by approximation. A similar result fails for non-separated algebraic spaces already when X is a non-separated valuation space, and, moreover, one can show that such spaces are “responsible” for the only obstruction for schematization of algebraic spaces and their morphisms. In the following result we construct such an obstruction. The inverse statement, which states that these are the only obstructions, is much more difficult and will be proved in [TT].

Proposition 5.1.7. *Let X be a qcqs algebraic space with a schematically dense quasi-compact pro-open subspace U , and let $X \rightarrow S$ be a morphism of qcqs algebraic spaces whose restriction to U is schematic. Assume that there exists a separated morphism $f: T \rightarrow X$ such that T is a valuation space, f maps the generic point to U , and the induced morphism $T \rightarrow S$ is not separated. Then X does not possess U -admissible modifications $X' \rightarrow X$ such that the morphism $X' \rightarrow S$ is schematic.*

Proof. Let $X' \rightarrow X$ be a U -admissible modification, and $T \rightarrow X$ be a separated morphism such that T is a valuation space whose generic point η is mapped to U . It suffices to show that if $X' \rightarrow S$ is schematic then T is S -separated. Let T' be the schematic image of η in $T \times_X X'$. Then $T' \rightarrow T$ is a modification of T ; hence $T' \xrightarrow{\sim} T$. Thus, the separated morphism $T \rightarrow X$ lifts to a separated morphism $T \rightarrow X'$, and therefore the composition $T \rightarrow S$ is separated by Lemma 5.1.6. \square

Remark 5.1.8. An analogous obstruction based on non-separated valuation spaces was (implicitly) used in [CT]. The main results of [CT] are that if k is a non-archimedean field then: (i) any étale equivalence relation $R \rightrightarrows U$ on Berkovich k -analytic spaces is effective whenever its diagonal is closed (and so U/R is a separated analytic space), (ii) any separated algebraic space over k is analytifiable by a k -analytic space (or a rigid space). In addition, it was shown by various examples that the separatedness assumptions (including closedness of the diagonal) cannot be ignored. The central mechanism of all those examples was a non-separated valuation space that obstructed analytification pretty similarly to the obstruction to schematization in the above Proposition. Since it was implicit in [CT], let us comment this briefly. Example [CT, 3.1.1] used valuations that correspond to analytic but non-rigid points, so it could be naturally interpreted in the analytic (but not rigid) geometry. Examples [CT, 5.1.3, 5.1.4] were based on height two valuations that correspond to non-analytic adic Huber’s point, so they could only be interpreted using the germ reduction functor or Huber’s adic spaces.

5.2. Valuative criteria.

5.2.1. *Decreasing the generic point of an SLP.* We will need the following lemma to establish Zariski local valuative criteria.

Lemma 5.2.1. *Let T be an SLP space, $\eta = \sqcup_{i=1}^r \eta_i$ be the scheme of its generic points, and $f: T \rightarrow X$ be a separated morphism of algebraic spaces such that $\eta \rightarrow X$ factors through a disjoint union of points $h: \eta' = \sqcup_{l=1}^m \eta'_l \rightarrow X$. Then there exists a unique SLP space T' whose scheme of generic points is η' , such that f factors into a composition of a surjective morphism $T \rightarrow T'$ and a separated morphism $T' \rightarrow X$ extending h . In addition, the construction of T' is étale-functorial, in the sense that it commutes with base changes corresponding to étale morphisms $\tilde{X} \rightarrow X$.*

Proof. The uniqueness of the factorization implies étale-functoriality. Thus, after replacing X with an affine presentation, we may assume that X is affine; set $X := \text{Spec}(A)$. Furthermore, it is sufficient to consider a single point η'_{l_0} and all connected components of T whose generic points factor through η'_{l_0} .

By Proposition 5.1.5, T is an affine SLP scheme, hence $T = \sqcup_{i=1}^r \text{Spec}(\mathcal{O}_i)$, where each \mathcal{O}_i is the intersection of finitely many valuation rings of $k(\eta_i)$, say $\mathcal{O}_i = \cap_{j=1}^{n_i} \mathcal{O}_{ij}$. Each intersection $\mathcal{O}'_{ij} := \mathcal{O}_{ij} \cap k(\eta')$ is a valuation ring of $k(\eta')$, hence $\mathcal{O}' := \cap_{ij} \mathcal{O}'_{ij}$ is a semi-local Prüfer ring with fraction field $k(\eta')$ by [Bou, Ch. VI, §7.1, Prop.1]. Note that $\mathcal{O}' = k(\eta') \cap \mathcal{O}_T(T) \subseteq k(\eta)$. Thus, the homomorphism $A \rightarrow \mathcal{O}$ factors through \mathcal{O}' , and we obtain the required factorization of f through $T' = \text{Spec}(\mathcal{O}')$. Obviously, T' is uniquely determined by the two conditions: T' is an affine SLP scheme with generic point η' , and the morphism $T \rightarrow T'$ is surjective. \square

5.2.2. *Zariski-local valuative criteria.* Assume that $f: Y \rightarrow X$ is a morphism between algebraic spaces. By an *SLP diagram of f* we mean an SLP space T with the space of generic points η , and a pair ρ of compatible separated morphisms $\rho_Y: \eta \rightarrow Y$ and $\rho_X: T \rightarrow X$. We say that such a diagram is *Zariski* if $\rho_Y: \eta \rightarrow Y$ is a disjoint union of Zariski points. Finally, if T is a valuation space then we say that ρ is a *valuative diagram of f* . The following proposition gives valuative criteria of properness and separatedness, which are Zariski local on the source:

Proposition 5.2.2. *Let $f: Y \rightarrow X$ be a morphism of algebraic spaces.*

- (i) *f is separated if and only if for any Zariski valuative diagram ρ , the morphism $\rho_X: T \rightarrow X$ admits at most one lifting $T \rightarrow Y$ compatible with $\rho_Y: \eta \rightarrow Y$,*
- (ii) *f is universally closed if and only if for any Zariski valuative diagram ρ , the morphism $\rho_X: T \rightarrow X$ admits a lifting $T \rightarrow Y$ compatible with $\rho_Y: \eta \rightarrow Y$,*
- (iii) *f is proper if and only if it is of finite type, and for any Zariski valuative diagram ρ , the morphism $\rho_X: T \rightarrow X$ admits unique lifting $T \rightarrow Y$ compatible with $\rho_Y: \eta \rightarrow Y$.*

Proof. Let us say that a class of SLP diagrams is *faithful* if it can be used in the valuative criteria similarly to (i), (ii), (iii). In particular, the proposition asserts that Zariski valuative diagrams form a faithful class. Our starting point is the usual valuative criteria [LMB, Th. 7.3, Prop. 7.8] that assert faithfulness of the class of valuative diagrams in which T is a scheme. Since local rings of SLP schemes are valuation rings, it follows that the class of all SLP diagrams with affine X is faithful. Our main tool to pass from one faithful class to another is the following assertion that follows from Lemma 5.2.1: if an SLP diagram $T \rightarrow X, \eta \rightarrow Y$ factors through an SLP diagram $T' \rightarrow X, \eta' \rightarrow Y$, and $T \rightarrow T'$ is surjective then there is a bijection between liftings $T' \rightarrow Y$ and $T \rightarrow Y$. In particular, since any SLP

space admits an affine presentation it follows that the class of all SLP diagrams is faithful. At this stage we already obtain the direct implications in (i), (ii), and (iii).

To prove the inverse implications (which are, actually, the criteria) we assume that f is not separated, universally closed, or proper, respectively. By the classical criterion this can be detected with a valuative diagram ρ which has an appropriate number of liftings $T \rightarrow Y$. By Lemma 5.2.1, ρ factors through a Zariski SLP diagram ρ' so that $T \rightarrow T'$ is surjective. Then T' is a valuation space, hence ρ' is a Zariski valuative diagram that has the same number of liftings as ρ . \square

Remark 5.2.3. If X is separated in the above criteria then T is separated, hence a valuation scheme. Already the case when X is a non-separated valuation space from §5.1.2, and Y is its generic point, shows that in general, one has to consider non-separated valuation spaces T in Zariski local valuative criteria. On the other hand, the condition that the morphism $T \rightarrow X$ is separated ensures that we consider only “minimally” non separated valuation spaces that are required.

5.2.3. Separated valuative diagrams. An SLP diagram $T \rightarrow X, \eta \rightarrow Y$ as above is called *separated* if its *diagonal* $\eta \rightarrow T \times_X Y$ is a closed immersion. It turns out that such diagrams suffice to test properness:

Theorem 5.2.4. *Let $f: Y \rightarrow X$ be a separated (resp. and finite type) morphism between qcqs algebraic spaces. Then f is universally closed (resp. proper) if and only if for any separated Zariski valuative diagram ρ the morphism $T \rightarrow X$ admits unique lifting $T \rightarrow Y$ compatible with $\eta \rightarrow Y$.*

Proof. In view of Proposition 5.2.2 we shall only prove that if any separated Zariski valuative diagram admits a lifting then the same is true for an arbitrary Zariski valuative diagram $T \rightarrow X, \eta \rightarrow Y$. The following lemma relates such a diagram to a separated Zariski diagram in a canonical way:

Lemma 5.2.5. *Let $T \rightarrow X, \eta \rightarrow Y$ be a valuative diagram. Then*

- (i) *There exists a maximal pro-open subspace $U \subseteq T$ such that the X -morphism $\eta \rightarrow Y$ extends to an X -morphism $i: U \rightarrow Y$. Furthermore, the extension i is unique and the space U is quasi-compact.*
- (ii) *Let η' be the closed point of U and $T' \subseteq T$ be the closure of η' equipped with the reduced subspace structure. Then $T' \rightarrow X, \eta' \rightarrow Y$ is a separated valuative diagram, where η' is mapped to Y via i .*

We will first finish the proof of the theorem. The separated valuative diagram $T' \rightarrow X, \eta' \rightarrow Y$ of the Lemma factors through a Zariski valuative diagram $T'' \rightarrow X, \eta'' \rightarrow Y$ by Lemma 5.2.1, and one easily sees that the latter diagram is also separated. A lifting $T'' \rightarrow Y$ exists by our assumptions, hence we obtain a lifting $T' \rightarrow T'' \rightarrow Y$. The latter lifting and the morphism $i: U \rightarrow Y$ induce a morphism $T \rightarrow Y$ because $T' \sqcup_{\eta'} U \xrightarrow{\sim} T$ by Proposition 4.3.6. Obviously, $T \rightarrow Y$ is an X -morphism that extends $\eta \rightarrow Y$, hence it is the required lifting, and we are done¹. \square

Proof of the lemma. (i) The uniqueness of i follows from the separatedness of the morphism $Y \rightarrow X$. Let U be the schematic image of the diagonal $\eta \rightarrow T \times_X Y$. Then $U \rightarrow T$ is a pro-open immersion by condition (4) of Corollary 4.1.6, and the projection $U \rightarrow X$ cannot be extended to any larger pro-open subspace of T .

¹One could show that $T' \sqcup_{\eta'} U \xrightarrow{\sim} T$ by a simple computation with presentations; so we referred to the theory of §3 only for the sake of convenience.

(ii) The diagonal $\eta' \rightarrow T' \times_X Y$ is the base change of the closed immersion $U \rightarrow T \times_X Y$ with respect to $T' \rightarrow T$. So, $(\eta', T') \rightarrow (Y, X)$ is a separated valuative diagram. \square

5.2.4. Uniformizable SLP spaces. We say that an SLP space X is *uniformizable* if the set η of its generic points is open. This happens if and only if $|X| \setminus \eta$ has no unbounded generalizing sequences $x_1 \prec x_2 \prec \dots$. Equivalently, any non-generic point of $|X|$ is a specialization of a codimension one point (note that any SLP space has finitely many such points). A typical example of uniformizable SLP spaces are those of finite height. Plainly, an SLP space is uniformizable if and only if one (hence any) of its étale coverings is so. An affine SLP space $X = \text{Spec}(A)$ is uniformizable if and only if there exists an element $\pi \in A$ such that the localization A_π coincides with the field of fractions of A . In this case, $A_\omega = \text{Frac}(A)$ if and only if ω is contained in all prime ideals of height one. If, moreover, A is a valuation ring then an element π as above is called a *uniformizer* (although, this contradicts the classical terminology when A is a DVR). We will use uniformizable SLP spaces and the following uniformizability criterion in [TT].

Lemma 5.2.6. *Let $f: Y \rightarrow X$ be a finite type morphism of algebraic spaces, and $T \rightarrow X$, $\eta \rightarrow Y$ be a separated Zariski SLP diagram of f . Then T is uniformizable.*

Proof. The schematically dominant pro-open immersion $\eta \rightarrow T$ is of finite type, since $\eta \rightarrow Y \times_X T$ and $Y \times_X T \rightarrow T$ are so. Hence it is open by Proposition 2.4.6. \square

Remark 5.2.7. What we call here a *uniformizable valuation ring* A (i.e. $\text{Spec}(A)$ is uniformizable) is classically called a microbial valuation ring, and a uniformizer $\pi \in A$ is sometimes called a microbe. Note that π is a uniformizer if and only if $\bigcap_{n \in \mathbf{N}} (\pi^n) = 0$. The (π) -adic completion \hat{A} does not depend on the choice of a uniformizer π and it is the only adic completion of A having injective completion homomorphism $A \rightarrow \hat{A}$. So, the only valuation rings that possess a reasonable completion theory are the uniformizable valuation rings.

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